Two Theoretical Research Questions Concerning the Structure of the Perfect Terrorist Cell

Jonathan David Farley

Abstract Two questions of theoretical interest regarding the number of cutsets in a poset are presented.

We model terrorist cells as *partially ordered sets*, or *posets*. For background, we refer the reader to Davey and Priestley 2002, Farley 2003 and Farley 2007, where some material from this article previously appeared.

The set of numbers $\{1,2,3,4\}$ is a poset; equivalently, we could be looking at four military officers, a captain, a major, a colonel, and a general. (We call such simply ordered posets *chains*.) In fact, as with numbers, the notation "$a \leq b$" is used to indicate that a is either the same person as b ($a = b$), or else a is a subordinate of b ($a < b$), albeit not necessarily an immediate subordinate.

Let z be a non-negative integer. A poset is z-ary if no member has more than z immediate subordinates. A *tree* is a connected poset whose diagram does not contain the "V" shape. Equivalently, a tree is a poset with a single leader (called the *root*) such that no member has more than one immediate superior. Thus the posets in Figure 1 are trees, but the poset of Figure 2 is not. The technical definition is that a connected poset is a *tree* if, for any member p, the set of superiors of p forms a chain.

Terrorist plans are formulated by the nodes at the top of the organization chart or poset (the leaders or *maximal* nodes); they are transmitted down via the edges (see Figures 1, 2, and 3) to the nodes at the bottom (the foot soldiers or *minimal* nodes), who presumably carry out those plans. The message, we assume, only needs to reach one foot soldier for damage to result. For example, suppose the poset represents a

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courier network. Only one messenger needs to succeed in parlaying the message; but the message must get through. We endeavor to block all routes from the maximal nodes (any one of those nodes) to the minimal nodes (any one of them) by capturing or killing some subset of the agents. Note that the agents we remove need not be maximal *or* minimal. Such a subset is called a *cutset*. A complete chain of command is called a *maximal chain* (not to be confused with a "maximal node").

If k terrorists are killed or captured at random, the probability we have found a cutset (and hence disrupted the cell, according to our model) is the number of cutsets of size k divided by the total number of subsets of size k.

We want to determine the structure of the perfect terrorist cell. Given all our assumptions, this translates into the following mathematical question: What z-ary connected partially ordered set with M maximal nodes and a total of n elements (members) has the fewest cutsets of size k?

It helps to use the following notation: Let Cutsets $(a_1, a_2, a_3, a_4, a_5, \dots)$ denote the number of cutsets of size $1, 2, 3, 4, 5, \dots$ respectively. The appendix lists all 63 five-member posets along with how many cutsets and minimal cutsets they have of each size. The labeling scheme indicates the number of superior-subordinate pairs ("non-trivial comparability relations") in a poset. For instance, the 5-member poset 6g in Figure 3 has 6 superior-subordinate pairs: ab, ac, bc, dc, ae, and de. (Note that this is different from the number of edges in the graph.) Starting from the n-member antichain, you can obtain all n-member posets by adding one superior-subordinate pair at a time (up to nC2, for the n member chain). For instance, Figure 4 shows that one can add a superior-subordinate pair to the 4-member poset 3c in three different ways. The extra pair is shown in bold in each of the three augmented posets. (The bold line between diagrams indicates that the number of cutsets is *decreasing* when you would expect it to increase. See below.)

In general, we merely list the posets you get by removing a superior-subordinate pair or by adding one. For instance, in the case of the 5-member poset 6g, by removing a pair you can get 5c, 5d, 5f, 5g, or 5h; by adding a pair you can get 7b, 7d, 7f, or 7h.

As an aside, note that in almost every case, the number of cutsets increases when a superior-subordinate pair is added. The cases where the numbers *decrease* are underlined for 4- and 5-member posets (i. e., for 5-member poset 6g we write "5c,5d,5f,5g,5h / 7b,7d,7f,7h").

We can write down all of the cases where the number of cutsets decreases when a superior-subordinate pair is added. For 4-member posets:

$$\begin{array}{c} 2\text{b} \rightarrow 3\text{c} \\ 3\text{c} \rightarrow 4\text{b} \end{array}$$

For 5-member posets:

$$\begin{array}{c} 2\text{b} \rightarrow 3\text{d} \\ 3\text{b} \rightarrow 4\text{b} \\ 3\text{b} \rightarrow 4\text{d} \\ 3\text{d} \rightarrow 4\text{f} \end{array}$$

3e → 4g
3e → 4i
4b → 5c
4d → 5c
4e → 5d
4e → 5g
4g → 5h
4i → 5h
5c → 6b
5d → 6g
5f → 6g
5g → 6g
5h → 6k
6c → 7b
6e → 7f
6h → 7d
6j → 7h
7b → 8b
7h → 8e

Problem 1. Let P and Q be n-member posets such that Q is obtained from P by the addition of one superior-subordinate pair. Let $(p_1, \ldots, p_n)$ and $(q_1, \ldots, p_n)$ be the cutset vectors of P and Q respectively. Can one characterize, in terms of the structures of P and Q, the situations where it is *not* the case that, for $1 \leq i \leq n, p_i \leq q_i$? Can one characterize the situations where, for $1 \leq i \leq n, p_i \geq q_i$?

Problem 2. Is the best z-ary poset with a single leader always a tree? If so, Campos, Chvátal, Devroye, and Taslakian 2007 have determined the structure of those trees.

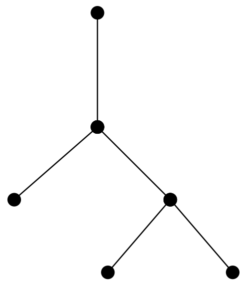


Figure 1(i). A binary tree.

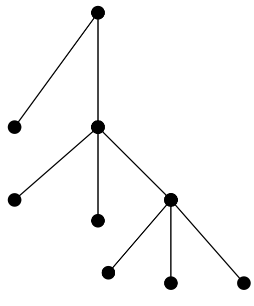


Figure 1(ii). A ternary tree that is not binary.

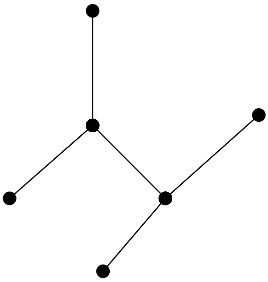


Figure 2. A poset that is not a tree.

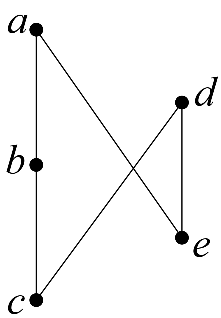


Figure 3. The 5-member poset 6g.

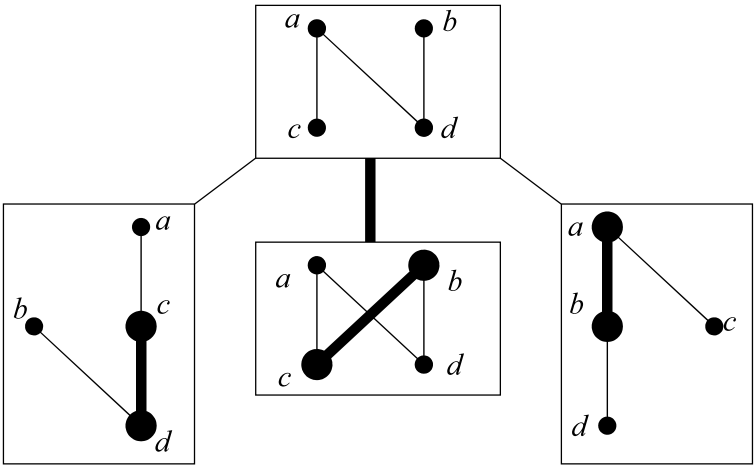


Figure 4. Adding superior-subordinate pairs to a poset.

Appendix: Cutsets and Minimal Cutsets of All n-Member Posets ($n \leq 5$)

$n = 1$

$n = 2$

0a

Min Cutsets (0,1)
Cutsets (0,1)
/ 1a

1a

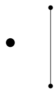
Min Cutsets (2,0)
Cutsets (2,1)
0a /

$n = 3$

0a

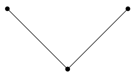
Min Cutsets (0,0,1)
Cutsets (0,0,1)
/ 1a

1a



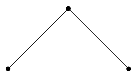
Min Cutsets (0,2,0)
Cutsets (0,2,1)
0a / 2a,2b

2a



Min Cutsets (1,1,0)
Cutsets (1,3,1)
1a / 3a

2b



Min Cutsets (1,1,0)
Cutsets (1,3,1)
1a / 3a

3a

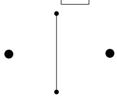
Min Cutsets (3,0,0)
Cutsets (3,3,1)
2a,2b /

$n = 4$

0a

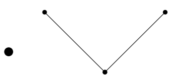
Min Cutsets (0,0,0,1)
Cutsets (0,0,0,1)
/ 1a

1a



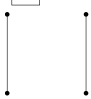
Min Cutsets (0,0,2,0)
Cutsets (0,0,2,1)
0a / 2a,2b,2c

2a



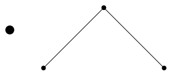
Min Cutsets (0,1,1,0)
Cutsets (0,1,3,1)
1a / 3a,3b,3c

2b



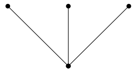
Min Cutsets (0,4,0,0)
Cutsets (0,4,4,1)
1a / 3c

2c



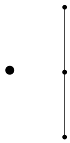
Min Cutsets (0,1,1,0)
Cutsets (0,1,3,1)
1a / 3b,3c,3d

3a



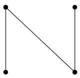
Min Cutsets (1,0,1,0)
Cutsets (1,3,4,1)
2a / 4a

3b



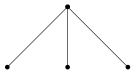
Min Cutsets (0,3,0,0)
Cutsets (0,3,3,1)
2a,2c / 4a,4c

3c



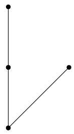
Min Cutsets (0,3,0,0)
Cutsets (0,3,4,1)
2a,2b,2c / 4a,4b,4c

3d



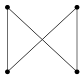
Min Cutsets (1,0,1,0)
Cutsets (1,3,4,1)
2c / 4c

4a



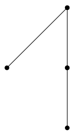
Min Cutsets (1,2,0,0)
Cutsets (1,5,4,1)
3a,3b,3c / 5a,5b

4b



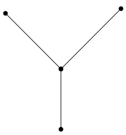
Min Cutsets (0,2,0,0)
Cutsets (0,2,4,1)
3c / 5a,5c

4c



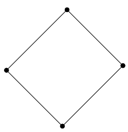
Min Cutsets (1,2,0,0)
Cutsets (1,5,4,1)
3b,3c,3d / 5b,5c

5a



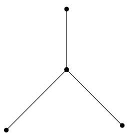
Min Cutsets (2,1,0,0)
Cutsets (2,6,4,1)
4a,4b / 6a

5b



Min Cutsets (2,1,0,0)
Cutsets (2,6,4,1)
4a,4c / 6a

5c



Min Cutsets (2,1,0,0)
Cutsets (2,6,4,1)
4b,4c / 6a

6a

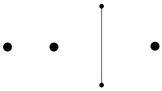
Min Cutsets (4,0,0,0)
Cutsets (4,6,4,1)
5a,5b,5c /

$n = 5$

0a

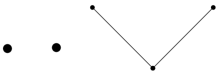
Min Cutsets (0,0,0,0,1)
Cutsets (0,0,0,0,1)
/ 1a

1a



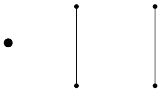
Min Cutsets (0,0,0,2,0)
Cutsets (0,0,0,2,1)
0a / 2a,2b,2c

2a



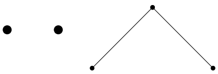
Min Cutsets (0,0,1,1,0)
Cutsets (0,0,1,3,1)
1a / 3a,3b,3c,3d

2b



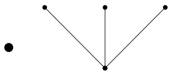
Min Cutsets (0,0,4,0,0)
Cutsets (0,0,4,4,1)
1a / 3b,3d,3e

2c



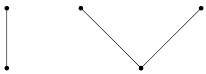
Min Cutsets (0,0,1,1,0)
Cutsets (0,0,1,3,1)
1a / 3c,3d,3e,3f

3a



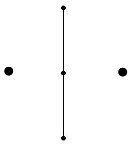
Min Cutsets (0,1,0,1,0)
Cutsets (0,1,3,4,1)
2a / 4a,4b,4c

3b



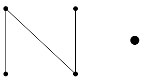
Min Cutsets (0,2,2,0,0)
Cutsets (0,2,7,5,1)
2a,2b / 4b,4d,4e

3c



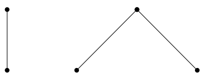
Min Cutsets (0,0,3,0,0)
Cutsets (0,0,3,3,1)
2a,2c / 4c,4e,4h

3d



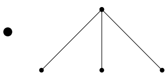
Min Cutsets (0,0,3,0,0)
Cutsets (0,0,3,4,1)
2a,2b,2c / 4b,4c,4d,4f,4g,4h,4i

3e



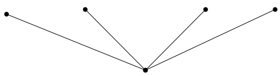
Min Cutsets (0,2,2,0,0)
Cutsets (0,2,7,5,1)
2b,2c / 4e,4g,4i

3f



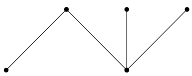
Min Cutsets (0,1,0,1,0)
Cutsets (0,1,3,4,1)
2c / 4h,4i,4j

4a



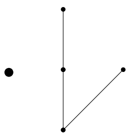
Min Cutsets (1,0,0,1,0)
Cutsets (1,4,6,5,1)
3a / 5a

4b



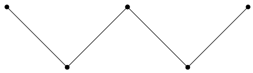
Min Cutsets (0,2,1,0,0)
Cutsets (0,2,6,5,1)
3a,3b,3d / 5a,5c,5d,5f

4c



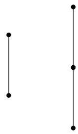
Min Cutsets (0,1,2,0,0)
Cutsets (0,1,5,4,1)
3a,3c,3d / 5a,5b,5d,5e,5f

4d



Min Cutsets (0,1,3,0,0)
Cutsets (0,1,6,5,1)
3b,3d / 5c,5g

4e



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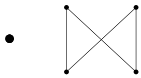
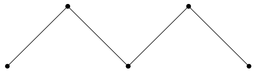
Cutsets (0,6,9,5,1)
3b,3c,3e / 5d,5g

4f



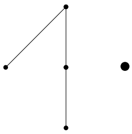
Min Cutsets (0,0,2,0,0)
Cutsets (0,0,2,4,1)
3d / 5b,5c,5h,5i

4g



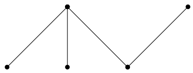
Min Cutsets (0,1,3,0,0)
Cutsets (0,1,6,5,1)
3d,3e / 5d,5h

4h



Min Cutsets (0,1,2,0,0)
Cutsets (0,1,5,4,1)
3c,3d,3f / 5e,5f,5g,5i,5j

4i



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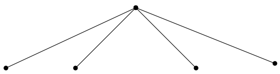
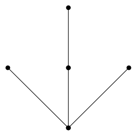
Cutsets (0,2,6,5,1)
3d,3e,3f / 5f,5g,5h,5j

4j



Min Cutsets (1,0,0,1,0)
Cutsets (1,4,6,5,1)
3f / 5j

5a



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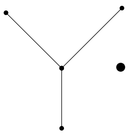

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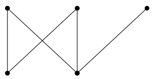

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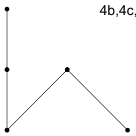

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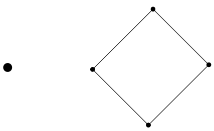

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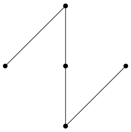
5f



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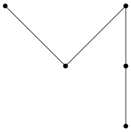

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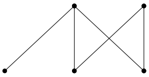
5h



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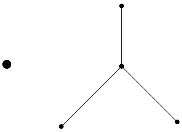
4f,4g,4i / 6e,6g,6k,6l

5i



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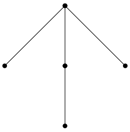
4f,4h / 6f,6h,6l

5j



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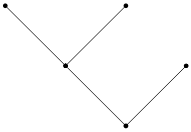
6a



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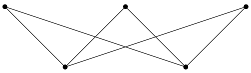
5a,5b,5c / 7a,7b,7c

6b



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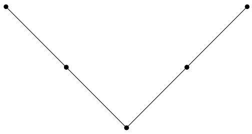
5c / 7a,7d

6c



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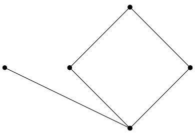
5a,5d / 7b

6d



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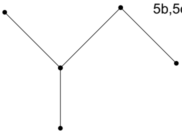
6e



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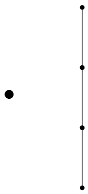
5b,5d,5f,5h / 7b,7f,7g

6f



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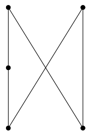
6g



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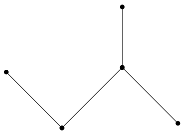
6h



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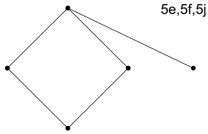
5c,5f,5g,5i / 7c,7d,7h

6i



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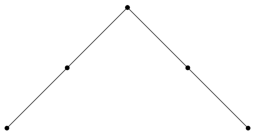
5e,5f,5j / 7e,7g,7h

6j



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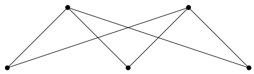
5g,5j / 7h

6k



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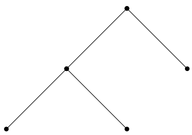
5h / 7f,7i

6l



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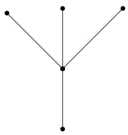
7a



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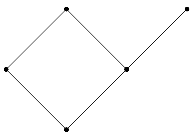
6a,6b / 8a

7b



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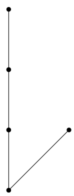
7c



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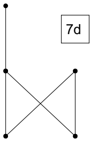
7d



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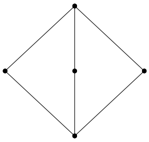
7e



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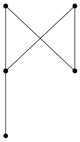
6d,6i / 8c

7f



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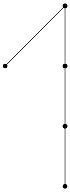
6e,6g,6k / 8b,8d,8f

7g



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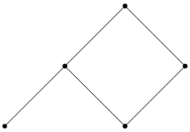
7h



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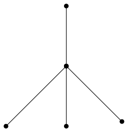
7i



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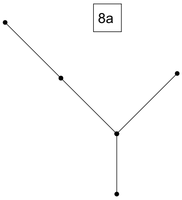
6k,6l / 8f

8a



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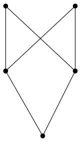
7a,7b,7c,7d / 9a,9b

8b



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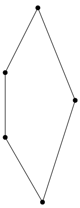
8c



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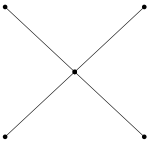

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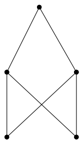

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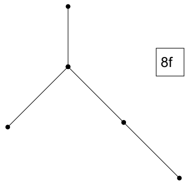

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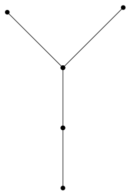
9a



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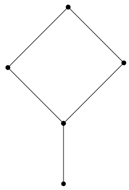
8a,8b,8d / 10a

9b



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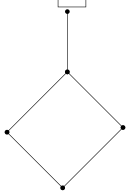

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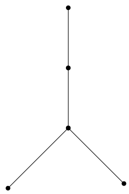

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8d,8e,8f / 10a

10a

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9a,9b,9c,9d /

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1. Victor Campos, Vašek Chvátal, Luc Devroye, and Perouz Taslakian, "Transversals in Trees" (2007).
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3. Jonathan David Farley, "Breaking Al Qaeda Cells: A Mathematical Analysis of Counterterrorism Operations (A Guide for Risk Assessment and Decision Making)," *Studies in Conflict and Terrorism* **26** (2003), 399–411.

4. Jonathan David Farley, *Toward a Mathematical Theory of Counterterrorism: Building the Perfect Terrorist Cell* (The Proteus Monograph Series, 2007).