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The Automorphism Group of the Fibonacci Poset: A "Not Too Difficult" Problem of Stanley from 1988

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Abstract. All of the automorphisms of the Fibonacci poset $Z(r)$ are determined ($r \in \mathbb{N}$). A problem of Richard P. Stanley from 1988 is thereby solved.

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1. Introduction—To be young again

Young's lattice Y, the set of Ferrers shapes partially ordered in a certain fashion, is a poset of tremendous combinatorial significance. As is well known, it is closely connected with the representation theory of the symmetric groups S_r ($r \in \mathbb{N}$).

In [9], Richard P. Stanley investigated lattices that share many of the interesting combinatorial properties of Y, the Fibonacci posets $Z(r)$ ($r \in \mathbb{N}$). These are posets whose elements are finite words generated from the alphabet $\{1_1, 1_2, \ldots, 1_r, 2\}$, where one defines the covering relation as follows: y covers x in $Z(r)$ if either

(1) $x = 2^k v$ and $y = 2^k 1_i v$, or
(2) $x = 2^k 1_i v$ and $y = 2^k 2v$

for some $i \in \{1, 2, \ldots, r\}$ and $v \in Z(r)$. In other words, one can either delete the first 1 or convert a 2 into a 1 (as long as no other 1's appear before it). See figures 1 and 3.

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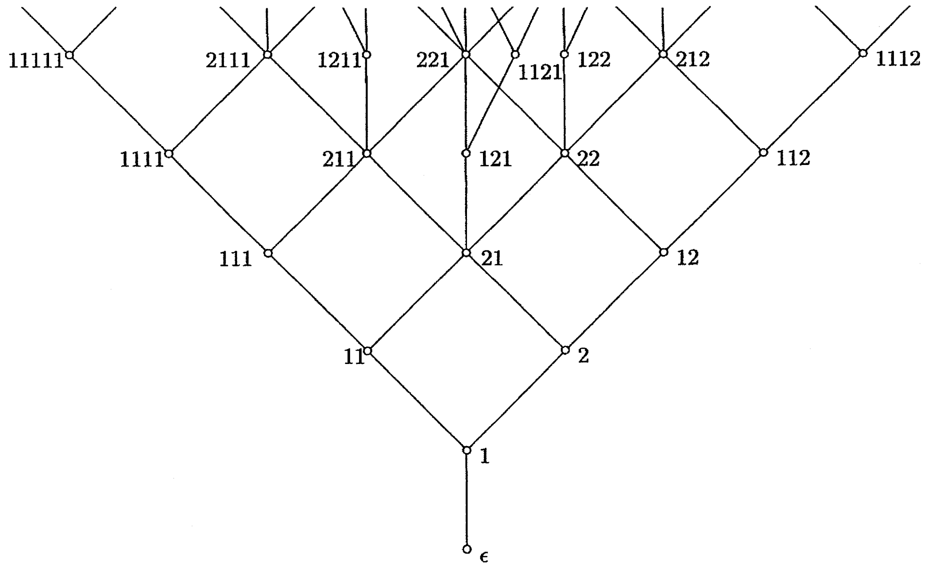

Figure 1. The Fibonacci poset $Z(1)$.

In [9], Problem 7, Stanley asks:

Problem ([9]) What is the automorphism group Aut $(Z(r))$ of the Fibonacci poset $Z(r)$?

(Admittedly, Stanley adds, "We suspect that Problem 7 should not be too difficult.")

We solve the problem by explicitly determining all of the automorphisms of $Z(r)$ (Theorem 4.5).

2. Definitions

Throughout this paper, r will denote a positive integer.

For basic terminology, see [2].

Let P be a poset; let $p, q \in P$. We say *q covers p* (denoted $p \lessdot q$) if $p < q$ and for all $r \in P$, $p \leq r < q$ implies $p = r$; p is a *lower cover* of q and q is an *upper cover* of p. An element is *join-irreducible* if it has exactly one lower cover; the set of such elements is denoted $\mathcal{J}(P)$.

A poset with a least element is *locally finite* if, for all $p \in P$, there are only finitely many elements of P below p. The *rank* of an element in such a poset is one less than the size of the largest chain whose top element is p.

In a lattice L, the least upper bound of $x, y \in L$ is denoted $x \vee y$.

An *automorphism* $\bar{\tau}$ of a poset P is an order-preserving bijection whose inverse is also order-preserving. Equivalently, if P is locally finite, a bijection $\bar{\tau} : P \to P$ is an automorphism if, for all $p, q \in P$, $p \lessdot q$ implies $\bar{\tau}(p) \lessdot \bar{\tau}(q)$ and vice versa.

Let ϵ denote the empty word. The *length* $|w|$ of a word w is the number of symbols in a reduced form of w.

If G and X are sets, G^X denotes the set of tuples $(g_x)_{x\in X}$ where $g_x \in G$ for all $x \in X$. Let S_r denote the symmetric group on r letters.

3. Facts about the Fibonacci poset $Z(r)$

The following facts come from [9], Section 5:

The Fibonacci poset $Z(r)$ is a locally finite modular lattice with least element ϵ. The rank of a word is the sum of its "letters." The lattice $Z(r)$ has the additional property that, if $w \in Z(r)$ has exactly k lower covers, then it has exactly $k + r$ upper covers. These facts make $Z(r)$ an *r-differential poset*.

Indeed, $Z(r)$ is the only r-differential (locally finite, modular) lattice such that every complemented interval has rank at most 2. It can be constructed inductively by "reflection" ([9], Section 6): One constructs $Z(r)$ rank by rank, reflecting the last layer one has built, then adding r new join-irreducible upper covers for each element [figures 2(a)–(c)].

One deduces that, for $Z(1)$, the number of elements of each rank is a Fibonacci number.

In [9], Section 6, Stanley observes that the symmetric group S_r acts on $Z(r)$.

Example 3.1 The 3-cycle $\sigma = (123)$ induces an automorphism $\bar{\sigma} : Z(3) \to Z(3)$ which sends, for instance, $w = 1_1 1_2 2 1_1 2 2 1_3$ to $\bar{\sigma}(w) = 1_2 1_3 2 1_2 2 2 1_1$.

"However," Stanley writes, "$Z(1)$ has (at least) an additional automorphism ω, defined by

$$\begin{aligned} \omega(v11) &= v2 \\ \omega(v2) &= v11 \\ \omega(w) &= w \quad \text{otherwise"} \end{aligned}$$

[i.e., if w is not of the form $v11$ or $v2$ for some $v \in Z(1)$]. The fact that ω is an automorphism is "obvious" by reflecting figure 1 about the vertical axis; but we prove it rigorously in Lemma 4.4.

Our list of references contains other papers dealing with the Fibonacci poset.

4. The automorphism group of the Fibonacci poset $Z(r)$: The solution to Stanley's problem

In this section we solve Stanley's problem (see Section 1) by finding all of the automorphisms of $Z(r)$ (Theorem 4.5).

The idea is that each word $w \in Z(r)$ has a set C_w of r join-irreducible upper covers; and an automorphism $\bar{\tau}$ of $Z(r)$ must send C_w to $C_{\bar{\tau}(w)}$. Hence, to each w we can associate an element of S_r. As $Z(r)$ is a locally finite lattice, the action of $\bar{\tau}$ on $\mathcal{J}(Z(r))$ determines $\bar{\tau}$.

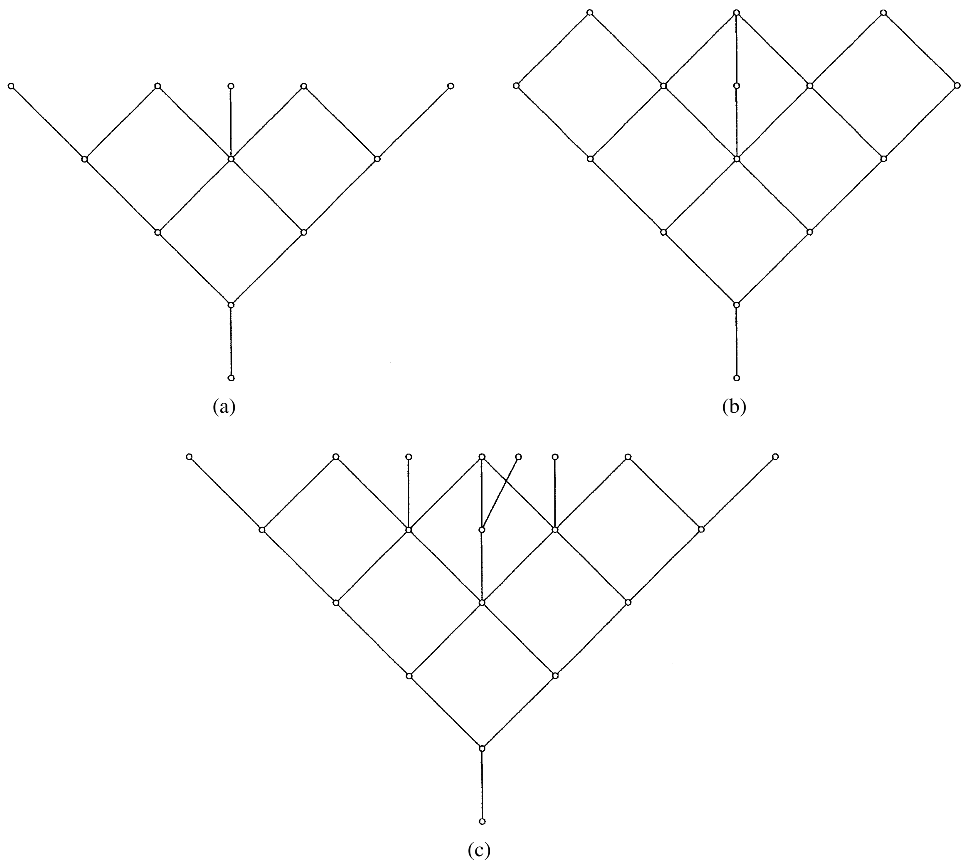

Figure 2. (a) Building $Z(r)$ by reflection. (b) Building $Z(r)$ by reflection. (c) Building $Z(r)$ by reflection.

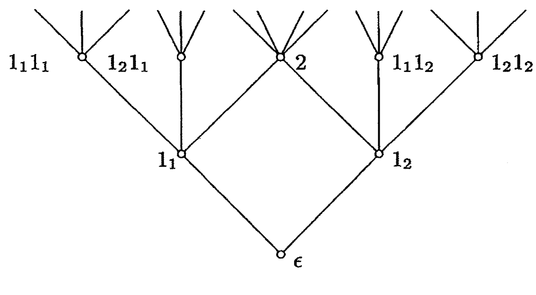

Figure 3. The Fibonacci poset $Z(2)$.

Lemma 4.1 (*Proposition* 2 *of* [4]) *The set $\mathcal{J}(Z(r))$ of join-irreducibles of $Z(r)$ is*

$$\{1_i v \mid 1 \le i \le r; v \in Z(r)\} \quad \text{if } r \ge 2,$$
$$\{1v \mid v \in Z(1)\} \cup \{2\} \quad \text{if } r = 1.$$

Proof: Clearly each element listed is join-irreducible. Now let $w \in \mathcal{J}(Z(r))$. Suppose w is not of the above form: then $w = 2v$ for some $v \in Z(r)$. If $r \ge 2$, then w has at least *two* lower covers, $1_1 v$ and $1_2 v$, a contradiction. Hence $r = 1$.

If $w = 21u$ for some $u \in Z(1)$, then w has the two lower covers $11u$ and $2u$; if $w = 22u$ for some $u \in Z(1)$, then w has the two lower covers $12u$ and $21u$. □

Lemma 4.2 *Let $\vec{\tau} = (\tau_w)_{w \in Z(r)}$ be a fixed element of $S_r^{Z(r)}$. Define a map $\bar{\tau} : Z(r) \to Z(r)$ inductively on the length of the argument, as follows: For all $w \in Z(r)$, let*

$$\bar{\tau}(w) = \begin{cases} 2^k & \text{if } w = 2^k \ (k \ge 0), \\ 2^k 1_{\tau_v(i)} \bar{\tau}(v) & \text{if } w = 2^k 1_i v \ [k \ge 0; 1 \le i \le r; v \in Z(r)]. \end{cases}$$

Then $\bar{\tau}$ belongs to Aut $(Z(r))$ *and $\bar{\tau}(2) = 2$.*

Proof: It is clear that $\bar{\tau}$ is a bijection whose inverse is $\bar{v}$ for some $\vec{v} \in S_r^{Z(r)}$. Thus it suffices to prove that if $x, y \in Z(r)$ and $x \lessdot y$, then $\bar{\tau}(x) \lessdot \bar{\tau}(y)$.

It is obvious that, for all $v \in Z(r)$, $\bar{\tau}(2v) = 2\bar{\tau}(v)$.

Case 1. $x = 2^k v$ and $y = 2^k 1_i v$ $[k \ge 0; 1 \le i \le r; v \in Z(r)]$

Then $\bar{\tau}(x) = 2^k \bar{\tau}(v)$ and $\bar{\tau}(y) = 2^k 1_{\tau v(i)} \bar{\tau}(v)$, so $\bar{\tau}(x) \lessdot \bar{\tau}(y)$.

Case 2. $x = 2^k 1_i v$ and $y = 2^k 2v$ $[k \ge 0; 1 \le i \le r; v \in Z(r)]$

Then $\bar{\tau}(x) = 2^k 1_{\tau_v(i)} \bar{\tau}(v)$ and $\bar{\tau}(y) = 2^k 2\bar{\tau}(v)$, so $\bar{\tau}(x) \lessdot \bar{\tau}(y)$. □

Let $\hat{\tau} \in$ Aut $(Z(r))$ be such that $\hat{\tau}(2) = 2$. Since $\hat{\tau}$ maps $\mathcal{J}(Z(r))$ bijectively onto itself, but fixes 2, then, by Lemma 4.1, for all $w \in Z(r)$ and $1 \le i \le r$, $\hat{\tau}(1_i w) = 1_j v$ for some $j \in \{1, 2, \ldots, r\}$ and $v \in Z(r)$. We will show that v depends only on w and not on i:

Lemma 4.3 *Let $\hat{\tau} \in$ Aut $(Z(r))$ be such that $\hat{\tau}(2) = 2$. For all $w \in Z(r)$, define $\tau_w \in S_r$ as follows: For $1 \le i \le r$, $\hat{\tau}(1_i w) = 1_{\tau_w(i)} \hat{\tau}(w)$.*

Let $\vec{\tau} = (\tau_w)_{w \in Z(r)}$. Define $\bar{\tau}$ as in Lemma 4.2.

Then $\hat{\tau} = \bar{\tau}$.

Proof: From what was said before the statement of the proposition, v must be $\hat{\tau}(w)$, as w is the unique lower cover of $1_i w$, so $\hat{\tau}(w)$ is the unique lower cover of $1_j v$, which is v. Thus $\tau_w \in S_r$ is well defined. □

Claim For all $v \in Z(r)$, $\hat{\tau}(2v) = 2\hat{\tau}(v)$.

Proof of Claim (by induction on $|v|$)**:** The claim follows from the definition of $\hat{\tau}$ *if* $v = \epsilon$.

Case 1. $v = 1_i u$ $[1 \leq i \leq r; u \in Z(r)]$.

Then $2u, 1_1 1_i u \lessdot 21_i u = 2v$, so $2u \vee 1_1 1_i u = 2v$. By the induction hypothesis, $\hat{\tau}(2u) = 2\hat{\tau}(u)$ and we know that $\hat{\tau}(1_1 1_i u) = 1_j 1_k \hat{\tau}(u)$ [where $1 \leq j, k \leq r$ and $\hat{\tau}(v) = 1_k \hat{\tau}(u)$]. Note that $2\hat{\tau}(u), 1_j 1_k \hat{\tau}(u) \lessdot 21_k \hat{\tau}(u)$, so $2\hat{\tau}(u) \vee 1_j 1_k \hat{\tau}(u) = 21_k \hat{\tau}(u)$ and hence $\hat{\tau}(2v) = 21_k \hat{\tau}(u) = 2\hat{\tau}(v)$.

Case 2. $v = 2u$ $[u \in Z(r)]$.

Then $21_1 u, 1_1 2u \lessdot 22u = 2v$, so $21_1 u \vee 1_1 2u = 2v$. By Case 1, $\hat{\tau}(21_1 u) = 2\hat{\tau}(1_1 u) = 21_j \hat{\tau}(u)$ for some $j \in \{1, 2, \ldots, r\}$. Also, $\hat{\tau}(1_1 2u) = 1_k \hat{\tau}(2u) = 1_k 2\hat{\tau}(u)$ for some $k \in \{1, 2, \ldots, r\}$.

Note that $21_j \hat{\tau}(u), 1_k 2\hat{\tau}(u) \lessdot 22\hat{\tau}(u)$, so $21_j \hat{\tau}(u) \vee 1_k 2\hat{\tau}(u) = 22\hat{\tau}(u)$ and hence $\hat{\tau}(2v) = 22\hat{\tau}(u) = 2\hat{\tau}(v)$, □

The lemma follows, as $\hat{\tau}$ and $\bar{\tau}$ agree on all words in $Z(r)$.

The following result is asserted without proof in [9], Problem 7.

Lemma 4.4 *The map* ω *of Section* 3 *belongs to* Aut $(Z(1))$.

Proof: As $\omega^2 = \mathrm{id}_{Z(1)}$, it suffices to prove that if $x, y \in Z(1)$ and $x \lessdot y$, then $\omega(x) \lessdot \omega(y)$.

Case 1. $x = 2^k v$ and $y = 2^k 1v$ $[k \geq 0; v \in Z(1)]$.

If $v = u11, u2$, or $u21$ for some $u \in Z(1)$, then $\omega(x) \lessdot \omega(y)$. If $v = 1$, then $\omega(x) = \omega(2^k 1) = 2^k 1 \lessdot 2^k 2 = \omega(2^k 11) = \omega(y)$.

Else, $v = \epsilon$. If $k = 0$, then $\omega(x) = \omega(\epsilon) = \epsilon \lessdot 1 = \omega(1) = \omega(y)$. If $k \geq 1$, then $\omega(x) = \omega(2^k) = 2^{k-1}11 \lessdot 2^{k-1}21 = 2^k 1 = \omega(2^k 1) = \omega(y)$.

Case 2. $x = 2^k 1v$ and $y = 2^k 2v$ $[k \geq 0; v \in Z(1)]$.

If $v = u11, u2, u21$, or ϵ for some $u \in Z(1)$,then $\omega(x) \lessdot \omega(y)$.

Else, $v = 1$, so $\omega(x) = \omega(2^k 11) = 2^k 2 \lessdot 2^k 21 = \omega(2^k 21) = \omega(y)$. □

Theorem 4.5 *The automorphism group of* $Z(1)$ *is isomorphic to* $\mathbb{Z}_2$, *the non-trivial automorphism being* $\omega : Z(1) \to Z(1)$ *defined for all* $w \in Z(1)$ *by*:

$$\omega(w) = \begin{cases} u2 & \textit{if } w = u11 \ [u \in Z(1)], \\ u11 & \textit{if } w = u2 \ [u \in Z(1)], \\ w & \textit{otherwise}. \end{cases}$$

The automorphism group of $Z(r)$ for $r \geq 2$ is in one-to-one correspondence with $S_r^{Z(r)}$. Given $\bar{\tau} = (\tau_w)_{w\in Z(r)} \in S_r^{Z(r)}$, define $\bar{\tau} : Z(r) \to Z(r)$ for all $w \in Z(r)$ by induction on $|w|$:

$$\bar{\tau}(w) = \begin{cases} 2^k & \text{if } w = 2^k\ (k \geq 0), \\ 2^k 1_{\tau_v(i)}\bar{\tau}(v) & \text{if } w = 2^k 1_i v\ [k \geq 0; 1 \leq i \leq r; v \in Z(r)]. \end{cases}$$

In general, Aut $(Z(r))$ *is not isomorphic to the product group $S_r^{Z(r)}$.*

Proof: If $\bar{\tau} \in$ Aut $(Z(1))$, then $\bar{\tau}(2)$ is either 2 or 11. By Lemmas 4.2 and 4.3, only the identity automorphism fixes 2, and $w^2 = \mathrm{id}_{Z(1)}$, so Aut $(Z(1)) = \{\mathrm{id}_{Z(1),\omega}\}$.

If $r \geq 2$, then $\bar{\tau}(2) = 2$ for all $\bar{\tau} \in$ Aut $(Z(r))$, since every other element of rank 2 is, by Lemma 4.1, join-irreducible. Hence the form of $\bar{\tau}$ follows from Lemmas 4.2 and 4.3.

Note that if $\bar{\tau}$ and $\bar{v}$ are automorphisms associated with the sequences $\vec{\tau} = (\tau_w)_{w\in Z(r)}$, $\vec{v} = (v_w)_{w\in Z(r)} \in S_r^{Z(r)}$, and $v \in Z(r)$ is a minimal element such that $\tau_v \neq v_v$ (say, $\tau_v(i) = j \neq k = v_v(i)$), then $\bar{\tau}(1_i v) = 1_j\bar{\tau}(v) \neq 1_k\bar{\tau}(v) = 1_k\bar{v}(v) = \bar{v}(1_i v)$. Hence, each automorphism constructed is unique.

To prove the last statement, note that $S_2^{Z(2)}$ has exponent 2. But consider the map $\bar{\tau}$: $Z(2) \to Z(2)$ where, for all $w \in Z(2)$,

$$\tau_w = \begin{cases} (12) & \text{if } w = \epsilon \textit{ or } w = 1_1, \\ (1)(2) & \text{otherwise.} \end{cases}$$

(See figure 3.)

Hence $\bar{\tau}(1_1 1_1) = 1_2 1_2$, $\bar{\tau}(1_2 1_2) = 1_2 1_1$, so $\bar{\tau}^2(1_1 1_1) \neq 1_1 1_1$ and $\bar{\tau}^2 \neq \mathrm{id}_{Z(2)}$. Therefore Aut $(Z(2)) \ncong S_2^{Z(2)}$. □

Collecting all the observations above, we can conclude as follows.

Corollary 4.6 *The automorphism group* Aut $(Z(r))$ *is isomorphic to $\mathbb{Z}_2 \wr S_r$.*

Proof: Let us denote the reflection of ω of $Z(1)$ and let $t_i = (i, i+1)(i = 1, \ldots, r-1)$ permutting the 1_i's. Then one can see that all the automorphisms of $Z(r)$ are the products of t_i and s under the following conditions:

order relations:

$$t_i^2 = s^2 = 1 \quad (i = 1, \ldots, r-1)$$

and the braid relations:

$$\begin{aligned} t_i t_j &= t_j t_i \quad (\text{for all } i, j = 0, \ldots, r-1 \text{ such that } |i-j| > 1, t_0 = s), \\ s t_j &= t_j s \quad (j = 2, \ldots, r-1), s t_1 s t_1 = t_1 s t_1 s, \\ t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1} \quad (i = 1, \ldots, r-1). \end{aligned}$$

Hence we can conclude that Aut $(Z(r)) \cong \mathbb{Z}_2 \wr S_r$ which is the Weyl group of type B_r. □

Remark According to the Shephard–Todd notation, this is $G(2, 1, r)$, which is a special case of the complex reflection group of type $G(d, 1, r) = \mathbb{Z}_d \wr S_r$ with $d = 2$.

For further detailed verifications of the relations of s and t_i in the proof above, please consult the second author.

The problem of Stanley from 1988 is thereby solved.

References

1. F. Brenti, "Log-concavity and combinatorial properties of Fibonacci lattices," *European Journal of Combinatorics* **12** (1991), 459–476.
2. B.A. Davey and H.A. Priestly, *Introduction to Lattices and Order*, 2nd edition, Cambridge University Press, Cambridge, 2002.
3. F.M. Goodman and S.V. Kerov, "The Martin boundary of the Young–Fibonacci lattice," *Journal of Algebraic Combinatorics* **11** (2000), 17–48.
4. R. Kemp, "Tableaux and rank-selection in Fibonacci lattices," *European Journal of Combinatorics* **18** (1997), 179–193.
5. D. Kremer, "A bijection between intervals in the Fibonacci posets," *Discrete Mathematics* **217** (2000), 225–235.
6. D. Kremer and K.M. O'Hara, "A bijection between maximal chains in Fibonacci posets," *Journal of Combinatorial Theory (A)* **78** (1997), 268–279.
7. S. Okada, "Algebras associated to the Young–Fibonacci lattice," *Transactions of the American Mathematical Society* **346** (1994), 549–568.
8. R.P. Stanley, "The Fibonacci lattice," *The Fibonacci Quartely* **13** (1975), 215–232.
9. R.P. Stanley, "Differential posets," *Journal of the American Mathematical Society* **1** (1988), 919–961.
10. R.P. Stanley, "Further combinatorial properties of two Fibonacci lattices," *European Journal of Combinatorics* **11** (1990), 181–188.