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# Distributive lattices of small width, II: A problem from Stanley's 1986 text *Enumerative Combinatorics*

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ABSTRACT

In Richard P. Stanley's 1986 text, *Enumerative Combinatorics*, the following problem is posed: Fix a natural number $k$. Consider the posets $P$ of cardinality $n$ such that, for $0 < i < n$, $P$ has exactly $k$ order ideals (down-sets) of cardinality $i$. Let $f_k(n)$ be the number of such posets. What is the generating function $\sum f_3(n)x^n$?
In this paper, the problem is solved.


## 1. Introduction

In the classic text *Enumerative Combinatorics* [6, p. 156], Stanley proposes the following problem: Fix a natural number $k$. Consider the posets $P$ of cardinality $n$ such that, for $0 < i < n$, $P$ has exactly $k$ order ideals (down-sets) of cardinality $i$. Let $f_k(n)$ be the number of such posets. What is the generating function $\sum f_3(n)x^n$? (In fact, Stanley first asks to enumerate the number $p(n)$ of such posets with the additional property that the only three-element antichains are the set of minimal elements and the set of maximal elements, and then asks to use the fact that $p(n) = 2^{n-7}$ for $n \geqslant 7$ to find $\sum f_3(n)x^n$. For more on this, see the end of Section 4.)

Fig. A.1(i) in Appendix A illustrates a poset with the above property; Table A.1 lists its order ideals.

Obviously $f_1(n) \equiv 1$ and it can be shown [7] that $f_2(n) = 2^{n-3}$ for $n \geqslant 3$. P. Edelman has made the simple observation that $f_k(n) = 0$ for $k > 3$ and $n > 1$, so it suffices to consider Stanley's problem for the case $k = 3$ [7, pp. 156, 177–178]. We observe that $f_3(0) = f_3(1) = 1$, $f_3(2) = 0$, $f_3(3) = 1$,

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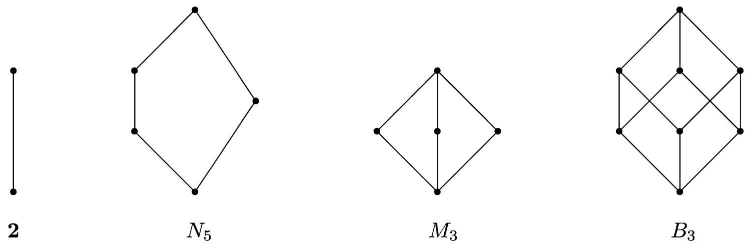


**Fig. 1.** Some commonly encountered posets.

$f_3(4) = 0$, $f_3(5) = 1$, $f_3(6) = 1$, $f_3(7) = 3$, and $f_3(8) = 6$. The posets illustrating these values of $f_3(n)$ for $n \geqslant 5$ are shown in Appendix A, Figs. A.2–A.5. (For $n = 0$, 1, or 3 the required poset is the $n$-element antichain.)

By Birkhoff's theorem [2, Theorem 5.12], Stanley's problem is equivalent to the following (for simplicity we exclude the trivial small cases): Given $n \geqslant 3$, how many distributive lattices of rank $n$ have the property that, for $0 < r < n$, there are exactly three elements of rank $r$? In this paper we investigate this problem, producing a system of recurrence relations that enumerate the isomorphism classes of these lattices (henceforward referred to as 3-*lattices*). We also derive a generating function from this system of recurrences.

While it may appear as if this is an isolated result, almost certainly the end of the subject, in fact it seems to relate to a problem posed by Ivo Rosenberg at the 1981 Banff Conference on Ordered Sets. Rosenberg asked to describe those posets that had the same width as their lattices of order ideals [5, p. 805]. The techniques developed in the first half of this paper should help address this other problem as well.

## 2. Basic concepts and notation

For notation or definitions not contained here, see [1] or [2].

A subset $D$ of a poset $P$ is a *down-set* or *order ideal* if, for all $d \in D$ and $p \in P$, $p \leqslant d$ implies $p \in D$.

If $a$ and $b$ are elements of a poset $P$, and $a \leqslant b$ or $b \leqslant a$, we say that $a$ and $b$ are *comparable*, and write $a \lesseqgtr b$. If $a < b$, and $a \leqslant x \leqslant b$ implies that $a = x$ or $x = b$, we write $a \prec b$ and say $b$ *covers* $a$; $a$ is a *lower cover* of $b$, and $b$ is an *upper cover* of $a$.

An element $a$ of a poset $P$ is *minimal* if there is no $b \in P$ such that $b < a$. Dually, $a$ is *maximal* if there is no $b$ with $a < b$. If $a \leqslant b$ for all $b \in P$ then we say that $a$ is a *least element* of $P$, and if $b \leqslant a$ for all $b$ we say it is a *greatest element*. The least and greatest elements of a given poset shall be denoted by 0 and 1, respectively.

We say $P$ is *connected* if the diagram of $P$ is connected as a graph.

For $a, b \in P$ with $a \leqslant b$, an *interval* in $P$ is a subposet of the form $[a, b] = \{x \mid a \leqslant x \leqslant b\}$.

A *chain* is a poset where every element is comparable to every other. The *rank* of a non-empty finite chain $C$ is $|C| - 1$. The *height* of an arbitrary non-empty finite poset $P$ is the rank of the longest chain contained in $P$. If all maximal chains in $P$ have the same rank, then that is the *rank* of $P$ and we say that $P$ is *graded* (or *ranked*). For any pair $a, b$ of minimal elements of a graded poset $P$, and every $c \in P$ with $c \geqslant a, b$, the intervals $[a, c]$ and $[b, c]$ have the same rank. We say that $b$ has rank $n$ in $P$ if the interval $[a, b]$ has rank $n$, where $a$ is minimal. We shall also use "rank" to refer collectively to the elements of a poset that have a given rank.

We say a lattice $L$ is *upper semimodular* if for every $x, y \in L$, $x \wedge y \prec x, y$ implies $x, y \prec x \vee y$. Dually, $L$ is *lower semimodular* if for every $x, y \in L$, $x, y \prec x \vee y$ implies $x \wedge y \prec x, y$. A lattice $L$ of finite height is modular if and only if it is both upper and lower semimodular. Lattices of finite height that are upper or lower semimodular are graded, and a lattice is modular if and only if it does not contain a sublattice isomorphic to $N_5$ (Fig. 1; see [1], Chapter II, Theorems 14 and 16). A lattice $L$ is *distributive* if for all $x, y, z \in L$ we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and dually. A lattice is distributive if and only if it does not contain a sublattice isomorphic to $N_5$ or $M_3$ (Fig. 1; in particular, distributive lattices are modular). Birkhoff's theorem says that a finite lattice is distributive if and only if it is isomorphic to the lattice of down-sets of a finite poset, ordered by set-inclusion.

## 3. Building the lattices

We begin by outlining a method for constructing all possible 3-lattices. The essence of the method is the idea of the *segment*.

**Definition.** Let $L$ be a 3-lattice (so $L$ has rank $n \geqslant 3$). A *segment* of $L$ is the set of all elements of $L$ that have a given rank $r < n$, and their upper covers, considered as a subposet of $L$. Any poset that can occur as a segment of a 3-lattice shall itself be referred to as a segment.

Since finite distributive lattices are graded, a segment simply consists of the elements of two consecutive ranks. We can thus view a 3-lattice as a stack of segments (consecutive segments in a stack intersect), and index the segments according to their height in the stack.

**Definition.** Let $S$ be a segment of a 3-lattice $L$. If $s$ is a minimal element of $S$, and $s$ has rank $r$ in $L$, then we say that $S$ has rank $r$ in $L$, and write $R_L^S = r$. If $T$ is a segment of $L$ with $R_L^T = R_L^S + 1$, then we say that $S$ *precedes* $T$, and $T$ *follows* $S$.

We now have basic building blocks we can use to construct 3-lattices, by divining which posets may occur as segments, and how they may relate to one another within a 3-lattice. Since the segments of rank 0 and $n-1$ of a rank $n$ 3-lattice each have only a single possible structure, only the "middle" segments shall be considered here.

**Lemma 3.1.** *If $L$ is a 3-lattice of rank $n$, and $S$ is a segment of $L$ with $0 < R_L^S < n-1$, then $S$ satisfies the following properties*:

(i) *$S$ has three minimal and three maximal elements.*
(ii) *Every element of $S$ is either minimal or maximal but not both.*
(iii) *For every $s \in S$, there exists a $t \in S$ such that $s \neq t$ and $s \leqslant t$.*
(iv) *For any distinct maximal $a, b \in S$, there is at most one $c \in S$ such that $c \prec a, b$, and dually (that is, for any distinct minimal $a, b \in S$, there is at most one $c \in S$ such that $a, b \prec c$).*

**Proof.** The first requirement follows from the width condition that there must be three elements of each rank (except the largest and smallest ranks). The rest are required by the fact that $L$ is a graded lattice. □

We may further restrict the candidates for segmenthood, using the following lemma.

**Lemma 3.2.** *Let $L$ be a 3-lattice of rank $n$, and let $S$ be a segment of $L$ such that $0 < R_L^S < n-1$. For any $a \in S$ such that $a$ is minimal, there exists $b \in S$ such that $b$ is minimal and $a, b \prec a \vee b$ in $L$.*

**Proof.** Suppose the contrary: that is, there exists a minimal $a \in S$ such that there is no minimal $b \in S$ with $a, b \prec a \vee b$ in $L$. This implies that any upper cover of $a$ has $a$ as its only lower cover. Let $b \in L$ be a minimal element of $S$ distinct from $a$, and let $c \in [a, a \vee b]$ be an upper cover of $a$ (by hypothesis, $c \neq a \vee b$). Clearly, $c \vee b = a \vee b$ in $L$. Furthermore, since $c$ has no lower covers other than $a$, we must have $c \wedge b = a \wedge b$. But then $\{a, b, c, a \wedge b, a \vee b\}$ is a sublattice of $L$ isomorphic to $N_5$, contradicting $L$'s distributivity. □

By inspection, we can see that Fig. 2 lists all the posets satisfying the conditions of Lemmas 3.1 and 3.2, together with the top and bottom segments. We can verify that all these candidates are in

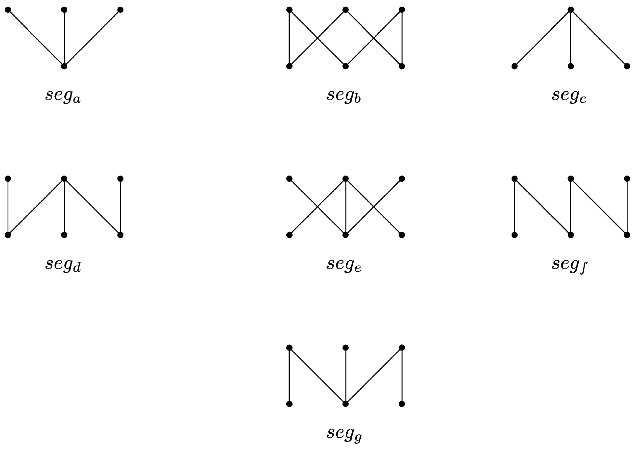


**Fig. 2.** Segment candidates.

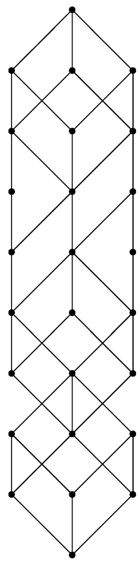

**Fig. 3.** A 3-lattice that contains all segment candidates.

fact segments by examining Fig. 3 [the distributive lattice corresponding to Fig. A.1(ii)]. By inspection, we see that this is a 3-lattice that has segments isomorphic to every candidate poset, and so each candidate is in fact a segment.

Now that we have a complete list of segments, we may proceed to show how these segments can be used to construct 3-lattices. First, we formulate rules describing which segments may follow a particular segment in a 3-lattice.

**Lemma 3.3.** *Let $S$ and $T$ be segments of a 3-lattice $L$. If $S$ has a minimal element $a$ with three upper covers, and $T$ follows $S$ in $L$, then $T$ is isomorphic to $seg_b$.*

**Proof.** Let $x, y, z$ be the maximal elements of $S$. These are the minimal elements of $T$ in $L$, and by modularity we must have $x \vee y, x \vee z, y \vee z \in T$. We prove by contradiction that these three elements are distinct. Assume, without loss of generality, that $x \vee y = y \vee z$ in $L$. (This implies $x \vee y = x \vee z =$

$y \vee z$.) But then $\{x, y, z, x \wedge z, x \vee z\}$ is a sublattice of $L$ isomorphic to $M_3$, which contradicts the fact that $L$ is distributive. Thus, $x \vee y, x \vee z$ and $y \vee z$ must be distinct elements of $T$. This condition is satisfied only by $seg_b$. □

**Corollary 3.4.** *Let $L$ be a 3-lattice, and let $S$ and $T$ be segments of $L$ with $R_L^T = R_L^S + 1$. If $S$ is isomorphic to $seg_a$, $seg_e$ or $seg_g$, then $T$ is isomorphic to $seg_b$. Dually, if $T$ is isomorphic to $seg_c$, $seg_d$ or $seg_e$, then $S$ is isomorphic to $seg_b$.*

**Proof.** The segments $seg_a$, $seg_e$ and $seg_g$ each have an element with three upper covers, and their duals are $seg_c$, $seg_e$ and $seg_d$, respectively ($seg_e$ and $seg_b$ are self-dual). □

We are using the fact that, in essence, there is a Boolean lattice "above" each element of a distributive lattice. The converse of the above lemma is also true.

**Lemma 3.5.** *Let $S$ and $T$ be segments of a 3-lattice $L$. If $T$ is isomorphic to $seg_b$, and $T$ follows $S$ in $L$, then there is a minimal element of $S$ which has three upper covers.*

**Proof.** Let $x, y, z$ be the maximal elements of $S$. We must have $x \wedge y, x \wedge z, y \wedge z \in S$. We will show by contradiction that all three elements are equal. Assume, without loss of generality, that $x \wedge y \neq y \wedge z$ in $S$. Then $y \wedge (x \vee z)$ is not well defined in $L$, since $x \wedge y, y \wedge z \leqslant y, x \vee z$ and $x \wedge y, y \wedge z \prec y$ and $x \wedge y \nleqslant y \wedge z$. Thus, we must have $x \wedge y = y \wedge z = x \wedge z$, implying that $x \wedge y$ has three upper covers in $S$. □

**Corollary 3.6.** *Let $L$ be a 3-lattice, and let $S$ and $T$ be segments of $L$ with $R_L^T = R_L^S + 1$. If $S$ is isomorphic to $seg_b$, then $T$ is isomorphic to $seg_c$, $seg_e$ or $seg_d$. Dually, if $T$ is isomorphic to $seg_b$, then $S$ is isomorphic to $seg_a$, $seg_e$ or $seg_g$.*

**Lemma 3.7.** *Let $S$ and $T$ be segments of a 3-lattice $L$, and let $T$ follow $S$. If $S$ has maximal elements $a, b$ such that $a \wedge b \notin S$, then $T$ is isomorphic to $seg_f$ or $seg_g$.*

**Proof.** Clearly, $a \vee b \notin T$. The only segments which have distinct minimal elements whose join is not in the segment are $seg_f$ and $seg_g$. □

**Corollary 3.8.** *Let $L$ be a 3-lattice, and let $S$ and $T$ be segments of $L$ with $R_L^T = R_L^S + 1$. If $S$ is isomorphic to $seg_d$ or $seg_f$, then $T$ is isomorphic to $seg_f$ or $seg_g$. Dually, if $T$ is isomorphic to $seg_g$ or $seg_f$, then $S$ is isomorphic to $seg_d$ or $seg_f$.*

We have now completely outlined the rules governing which segments may follow a particular segment within a 3-lattice. These rules are illustrated in Fig. 4; each segment is connected by an arrow to each segment that may follow it in a 3-lattice. For example, $seg_b$ and $seg_e$ can each follow the other, and so there is a double arrow between the two boxes. Similarly, $seg_f$ can follow itself, represented by a looping arrow. However, $seg_g$ cannot follow $seg_b$ (by 3.6), so there is no arrow from the $seg_b$ box to the $seg_g$ box.

We shall see that these rules are (almost) sufficient for a poset constructed of segments to be a 3-lattice, but first we must formalize what we mean by "construction."

**Definition.** Let $P$ and $Q$ be finite posets with disjoint underlying sets, and let $P_{\max}$, $Q_{\min}$ be the sets of maximal elements of $P$ and minimal elements of $Q$, respectively. If there is a bijection $\phi : P_{\max} \to Q_{\min}$, we define the *$\phi$-concatenation* of $P$ and $Q$ to be the poset with underlying set $R = P \cup Q - Q_{\min}$, and partial order $a \leqslant b$ in $R$ if and only if one of the following holds:

(i) $a, b \in P$ and $a \leqslant b$ in $P$.

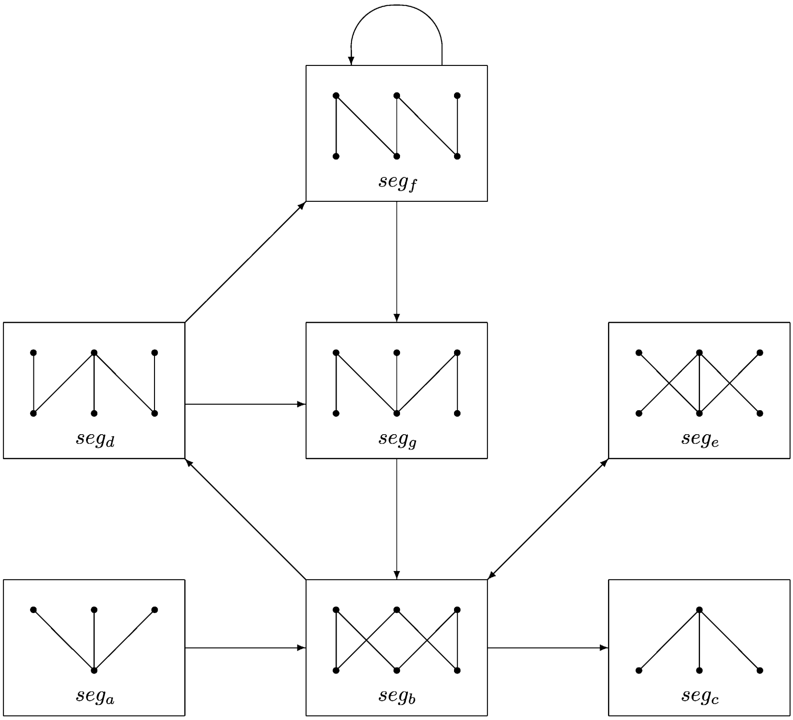


**Fig. 4.** A flow diagram illustrating how segments may follow one another in a 3-lattice.

(ii) $a, b \in Q$ and $a \leqslant b$ in $Q$.
(iii) $a \in P$, $b \in Q$, and there exists $c \in P_{\max}$ such that $a \leqslant c$ in $P$ and $\phi(c) \leqslant b$ in $Q$.

If $R$ is the $\phi$-concatenation of $P$ and $Q$ we write $R = P \&_\phi Q$ (always so that the poset which contains the domain of $\phi$ occurs first). Concatenating simply means gluing the poset diagrams according to $\phi$. We shall refer to functions of the above type as *concatenation functions*.

This is the tool we shall use to construct 3-lattices from their component segments. In the interest of brevity, we will often speak of concatenating one poset onto another, without specifying a function between them.

In constructing a 3-lattice by repeated concatenation of segments, we may not always choose our concatenation function freely.

**Lemma 3.9.** *Suppose $S$ and $T$ are segments of a 3-lattice $L$ such that*

(i) *There exist maximal elements $s_1$ and $s_2$ of $S$ which do not have a meet in $S$.*
(ii) *There exist minimal elements $t_1$ and $t_2$ of $T$ which do not have a join in $T$.*
(iii) *$T$ follows $S$ in $L$.*

*Let $\phi : S_{\max} \to T_{\min}$ be a concatenation function. If $S \&_\phi T$ is isomorphic to $S \cup T$ as a subposet of $L$, then $\phi(\{s_1, s_2\}) = \{t_1, t_2\}$.*

**Proof.** Assume $\phi(\{s_1, s_2\}) \neq \{t_1, t_2\}$, and denote the remaining minimal element of $T$ by $t_3$. Clearly, $\phi(\{s_1, s_2\}) \cap \{t_1, t_2\}$ cannot be empty, so set $\phi(s_1) = t_1$, $\phi(s_2) = t_3$. By Lemma 3.2, $t_1 \vee t_3$ exists in $T$,

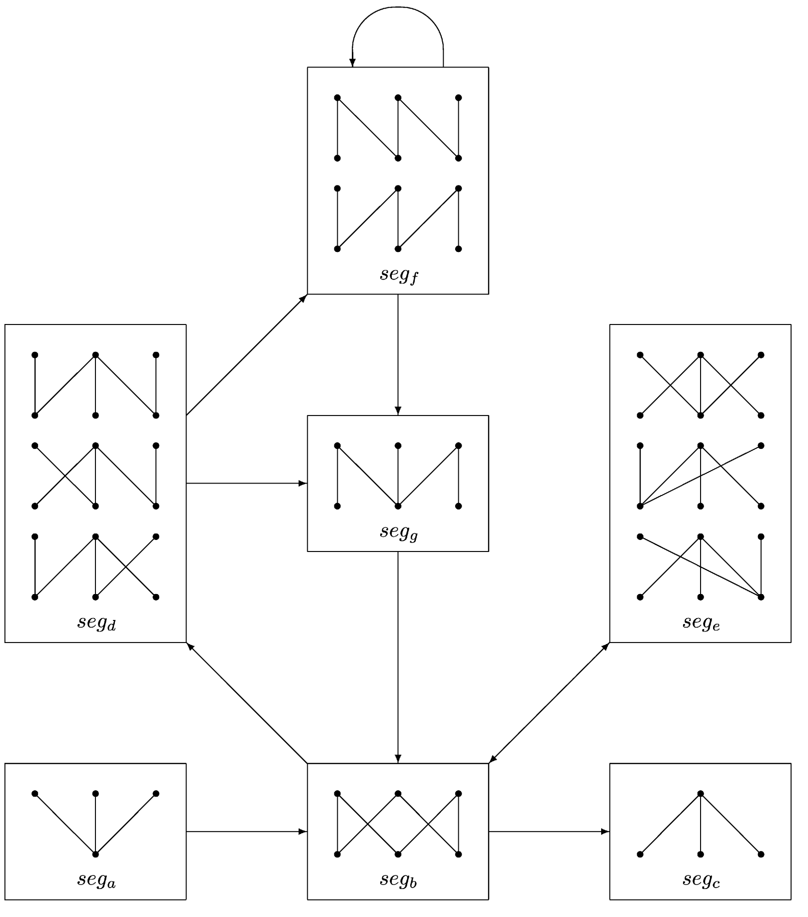


**Fig. 5.** A flow diagram summarizing the rules for 3-lattice construction.

and so $s_1 \vee s_2$ exists in $L$, and covers $s_1$ and $s_2$. However, the modularity of $L$ implies $s_1 \wedge s_2 \prec s_1, s_2$, contradicting our definition. □

This now completes our list of requirements for the construction of 3-lattices. We shall denote posets that comply with these rules as follows:

**Definition.** A *stack* is a finite poset constructed via a path in the finite state diagram of Fig. 5. If it starts with $seg_a$ and ends with $seg_c$, it is a *complete stack*. Note that a stack is ranked, and (while we do not need this fact) the dual of a stack is a stack.

We illustrate the rules of construction in Fig. 5 in a manner similar to Fig. 4. Arrows indicate permissible concatenations, and the various arrangements of a given segment represent the different choices of concatenation function which may be used. That is, the relative positions of each minimal element in a segment determine where the function sends each maximal element of the preceding segment: the leftmost element goes to the leftmost element, the middle element goes to the middle element, and the rightmost to the rightmost. One can imagine concatenating a segment onto a stack so that it does not appear in any of the forms given; however, we can always transform it into one of these forms by rearranging the maximal elements.

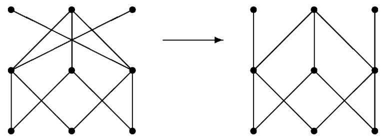

**Fig. 6.** Rearranging a segment.

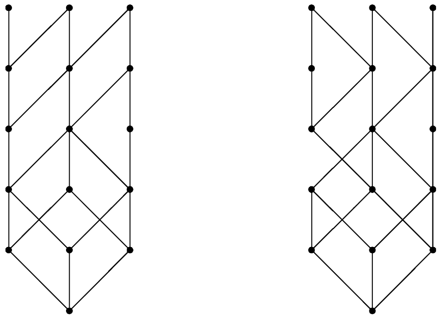

**Fig. 7.** Stacks constructed using different arrangements of $seg_d$ and $seg_f$.

An example of this is given in Fig. 6. We switch the maximal elements left for right to obtain a given arrangement of $seg_d$.

In Appendix A, we give many examples of posets constructed using these rules (Figs. A.1–A.5). The corresponding "words" should be self-explanatory; for instance, the letter $e_0$ indicates the top drawing of $seg_e$, $e_1$ the second, and $e_2$ the third. The complete stacks correspond to words in the regular language

$$ab\big\{\big\{\{e_0, e_1, e_2\}b\big\}^*\big\{\{d_0, d_1, d_2\}\{f_0, f_1\}^* gb\big\}^*\big\}^* c.$$

In Fig. 7, we give an example of stacks constructed using different arrangements of segments. In the first stack, we have $seg_b$ followed by $seg_d$ in its first arrangement, whereas in the second stack $seg_d$ is in its second arrangement. In this case we can see that the two arrangements of $seg_d$ are equivalent in a sense, since we can rearrange one to obtain the other. This is not true of the different arrangements of $seg_f$ which occur in the top ranks, however. The question of when different arrangements (and hence choices of concatenation function) are equivalent is key in the enumeration of the isomorphism classes of 3-lattices.

It is clear from the preceding lemmas that every 3-lattice must necessarily be a complete stack. But is every complete stack a 3-lattice? That is, is every complete stack a distributive lattice? The issue here is that it is not clear a priori that a poset like that in Fig. A.1(iii) is even a lattice, much less a distributive one. The poset on the right of Fig. A.6, for instance, is built from two $seg_b$ segments, so that superficially it resembles the distributive lattice $\mathbf{2}^3$ on the left; but is it a (distributive) lattice? (Find the answer in Section 5.)

**Corollary 3.10.** *For all $a, b, c$ in a stack $P$, if $a, b \prec c$, and $a, b$ are not minimal, then there is some $d \in P$ such that $d \prec a, b$, and dually.*

**Proof.** This follows from Fig. 5. □

This property can be shown to hold in any interval of a stack as well.

**Lemma 3.11.** *Let $P$ be a stack, and let $I$ be an interval in $P$. If $a, b \in I$ are distinct and have a common upper cover in $I$, then they have a common lower cover in $I$, and dually.*

**Proof.** Assume $a$ and $b$ have a common upper cover in the interval. If $c$ is a maximal lower bound of $a, b$ in $I$ ($c$ exists, since $I$ has a least element), then we must have $a'$ and $b'$ such that $c \prec a' \leqslant a$, and $c \prec b' \leqslant b$.

Suppose, by way of contradiction, that $a', b'$ are distinct from $a, b$. Then we must have $d, d' \in P$ such that $d \prec a, b$ and $a', b' \prec d'$. The maximality of $c$ in $I$ implies that the intervals $[a', a]$ and $[b', b]$ are disjoint, and $d' \nleqslant a, b$. If we suppose that $a'$ and $b'$ have rank $n$ in $P$, we may conclude that any element having rank $> n$ must be greater than $c$, since the three elements of rank $n+1$ are $d'$, an upper cover of $a'$, and an upper cover of $b'$. When $a', b' \nprec a, b$, this implies $c \leqslant d$ (and hence $d \in I$), contradicting $c$'s maximality.

When $a' \prec a$ and $b' \prec b$, $\{a, a', b, b', d, d'\}$ is a segment of $P$ isomorphic to $seg_b$ (since $seg_b$ does not occur inside any other segment). By Fig. 5, only $seg_a$, $seg_g$ and $seg_e$ can precede a $seg_b$ in a stack, and each of these segments has the property that any element with two distinct upper covers must in fact have three such covers. Thus, $c \leqslant d$ as before. The dual follows by a similar argument. □

We now proceed to show that every complete stack is indeed a 3-lattice, using the property above, along with the following proposition.

**Proposition 3.12.** *(See [4, Theorem 5.2].) Let $P$ be a graded poset with $0$ and $1$, and rank at least $3$. If every rank $3$ interval is a distributive lattice, and if, for every interval of rank at least $4$, the interval minus its endpoints is connected, then $P$ is a distributive lattice.*

We show that complete stacks satisfy the two conditions of this proposition in the following lemmas.

**Lemma 3.13.** *Let $P$ be a stack, and let $I = [a, b]$ be some interval in $P$. If $I$ has a rank greater than $2$, $I - \{a, b\}$ is connected.*

**Proof.** By Lemma 3.11, for every pair $x, y \in I$ of distinct elements such that $a \prec x, y$, there exists $z \in I$ such that $x, y \prec z$. Since the rank of $I$ is greater than 2, $b$ does not cover any upper cover of $a$. Thus, any two minimal elements of $I - \{a, b\}$ are connected through their common upper cover, and by definition every non-minimal element is greater than, and hence connected to, some minimal element. □

**Lemma 3.14.** *Let $P$ be a stack. If $I = [a, b]$ is an interval in $P$ having rank $3$, then $I$ is a distributive lattice.*

**Proof.** Clearly, $a$ can have no fewer than one and no more than three upper covers in $I$ (and dually for $b$). Suppose $a$ has three upper covers in $I$, denoted by $a_1$, $a_2$ and $a_3$. By Lemma 3.11, $a_1$ and $a_2$, $a_2$ and $a_3$, and $a_3$ and $a_1$ have common upper covers in $I$, and by Fig. 5 these are all distinct. Thus, $b$ must have three lower covers, and $I \cong B_3$.

Now suppose $a$ has just two upper covers in $I$, $a_1$ and $a_2$, and let $a_1, a_2 \prec c$ in $I$. If $b$ has three lower covers in $I$, then $a$ must have three upper covers in $I$ by the dual of the previous argument, contradicting our hypothesis. If $b$ has just one lower cover in $I$, $b_1$, then $c = b_1$, and $I$ is isomorphic to the first lattice in Fig. 8. If $b$ has exactly two lower covers in $I$, $b_1$ and $b_2$, then we must have $c = b_1$ or $c = b_2$, so assume the former. Now $b_2$ covers at least one of $a_1, a_2$, and by Fig. 5 we cannot have $a_1, a_2 \prec b_2$. Thus $I$ is isomorphic to the second lattice in Fig. 8, which is again distributive.

Finally, we consider the case where $a$ has exactly one upper cover in $I$. If $b$ has exactly two lower covers in $I$, we have the dual of the first part of the previous case. If $b$ has only one lower cover in $I$, then $I$ is just the 4-element chain. Thus $I$ is distributive in every possible case. □

We may now apply Proposition 3.12.

**Proposition 3.15.** *A poset $L$ is a $3$-lattice if and only if it is a complete stack.*

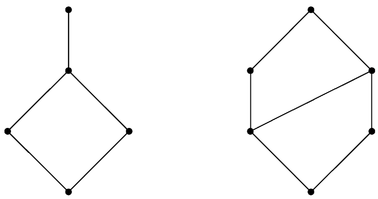

**Fig. 8.** Possible rank 3 intervals of a stack.

**Proof.** As already stated, every 3-lattice is a complete stack. By Lemmas 3.13 and 3.14, and by Proposition 3.12, we see that every complete stack is a distributive lattice (since complete stacks must have rank at least 3). Clearly, every stack satisfies the requirement on the number of elements of each rank, so we conclude that, if $L$ is a complete stack, it is a 3-lattice. □

## 4. Isomorphism classes of 3-lattices

In this section, we will determine the criteria that distinguish distinct (non-isomorphic) 3-lattices from one another, and enumerate the isomorphism classes. It is easily seen that two 3-lattices are isomorphic only if they have the same rank, and if segments of corresponding rank are isomorphic (that is, for a given rank $n$, the segment of rank $n$ in one 3-lattice is isomorphic to the segment of rank $n$ in the other). These conditions are not sufficient, and we must do more work before we may begin to enumerate the isomorphism classes.

Our main difficulty arises from the fact that, for each segment concatenation we perform in the construction of a 3-lattice, we may be able to choose a number of different concatenation functions. As alluded to in our discussion of Fig. 7, different choices of $\phi$ in stacks which are otherwise identical may or may not produce 3-lattices that are isomorphic. (See Figs. A.4(ii), A.5(ii), A.5(iii), and A.5(v), for instance.)

An example illustrating this difficulty is the following: In Fig. A.7(i), we see three different drawings of the same poset, the 6-element crown. In Fig. A.7(ii), they are "stacked" in three different ways, to get three diagrams of posets. Two of them are isomorphic. Which two? (This example is like the illusion of the two faces and the vase: one might see it immediately, but, if not, see Section 5 for the answer.)

In order to resolve this problem, we shall use a recursive approach. We determine how many ways we can concatenate a given segment onto a stack so as to yield distinct stacks, and thus express the number of stacks of a given rank in terms of the number having rank one less. We begin with the following definition:

**Definition.** Let $S$ be a segment, and let $s$ be an element of $S$ with rank $n$. We say that $s$ is a *key element* of $S$, or simply a *key*, if $s$ is the only element of rank $n$ to have $c$ lower covers in $S$ (where $1 \leqslant c \leqslant 3$). We also say $s$ is a *key* of $S$ if $s$ is the only element of rank $n$ to have $c$ upper covers in $S$ (where $1 \leqslant c \leqslant 3$).

Referring back to Fig. 5, we can see that all segments, with the exception of $seg_a$, $seg_b$ and $seg_c$, have key elements in both their top and bottom ranks. The term "key" is used because each key element is unique in its rank.

Since a stack is a concatenation of segments, we can speak of key elements in stacks.

**Definition.** We shall say that an element of a stack $S$ is a *key element* if it is a key element of some segment of $S$. If $k$ is a key element of $S$ with rank $n$ in $S$, and $k$ is a key element of the segment of rank $n-1$ in $S$, we say that $k$ is an *upper key*. Similarly, if $k$ has rank $n$ and is a key element of the segment of rank $n$ in $L$, we shall say that $k$ is a *lower key*.

Thus, a key element is a lower key if it is a minimal key element of some segment, and dually for upper keys. Note that, if $S$ is a stack of rank $n$, there is at least one key element of rank $r$ for $2 \leqslant r \leqslant n-2$. There may be more than one key element of a given rank in a stack, since the maximal key of one segment is not necessarily sent to the minimal key of the segment following it when the two are concatenated. There can, however, be at most one lower (or upper) key of each rank.

The importance of key elements derives from the following simple lemma.

**Lemma 4.1.** *Let $P$ and $Q$ be stacks, and let $f$ be an isomorphism from $P$ to $Q$. If $p \in P$ and $q \in Q$ are lower keys of rank $n$, then we must have $f(p) = q$. If $p$ and $q$ are upper keys of rank $n$, then $f(p) = q$. If $p \in P$ is a non-key element of rank $n$, then $f(p)$ is a non-key element of rank $n$ in $Q$.*

In Fig. 5, we see that $seg_f$ and some arrangements of $seg_d$ and $seg_e$ appear asymmetrical, with key elements off to one side. As we shall see, such asymmetries induce an orientation on the stacks in which they occur (or portions thereof).

**Definition.** A stack $S$ is said to have a *twist of rank $n$* if one of the following holds:

(i) There are two distinct key elements of rank $n$ (one lower key, one upper key).
(ii) There is a key element of rank $n-1$ which is covered by exactly one non-key element (but may have other upper covers).

By examining Fig. 5 we can see that twists occur wherever a segment appears asymmetrical, roughly speaking. Twists of the first type occur when a $seg_f$ follows a $seg_d$, or a second $seg_f$ with the same orientation, or when $seg_g$ follows $seg_f$. Twists of the second type occur when an asymmetric $seg_d$ or $seg_e$ follows a $seg_b$ and the $seg_b$ is preceded by $seg_e$ or $seg_g$, or when two $seg_f$'s of the opposite orientation occur together, or $seg_g$ follows $seg_f$. (We are not saying these are the only ways twists can occur.)

The importance of twists is that they are fixed in a certain sense under automorphism.

**Lemma 4.2.** *Let $L$ be a stack, and suppose $L$ has a twist of rank $n$. If $f$ is an automorphism on $L$, and $a \in L$ is of rank $n$, then $f(a) = a$.*

**Proof.** For twists of the first type, this is a trivial consequence of Lemma 4.1. For twists of the second type, suppose $k$ is a key element of rank $n-1$ with a single non-key upper cover, $\tilde{k}$. So $f(k) = k$, which implies $f(\tilde{k}) = \tilde{k}$. By again using Fig. 5 and Lemma 4.1, we see that the remaining elements of rank $n$ must also remain fixed under $f$. □

Indeed, it can be shown that, essentially, twists fix all elements which occur above them, provided the stacks in which they occur are not of the following form:

**Definition.** A stack $P$ is said to be a *switch stack* if it satisfies one of the following:

(i) $P$ has no twists.
(ii) There exists some segment $S$ in $P$ such that $S$ has rank $n$ in $P$, $S$ is isomorphic to $seg_e$, and $P$ has no twists of rank strictly greater than $n$.

Referring back to Fig. 5, we see that a stack is a switch stack if it contains no segment isomorphic to $seg_f$, and every occurrence of $seg_d$ is symmetrical, or if there is a $seg_e$ such that this is true in the portion of the stack above it. (We are not claiming this is "if and only if.")

Intuitively, if $P$ is a switch stack, we can "flip" around any twist that is induced by the concatenation of a segment onto $P$, showing that the two possible twisted forms of the concatenation are in fact equivalent. This is straightforward when $P$ has no twists. In the case where $P$ has twists, but there is a $seg_e$ in $P$ such that no twists follow it, we can flip the portion of the stack that follows it,

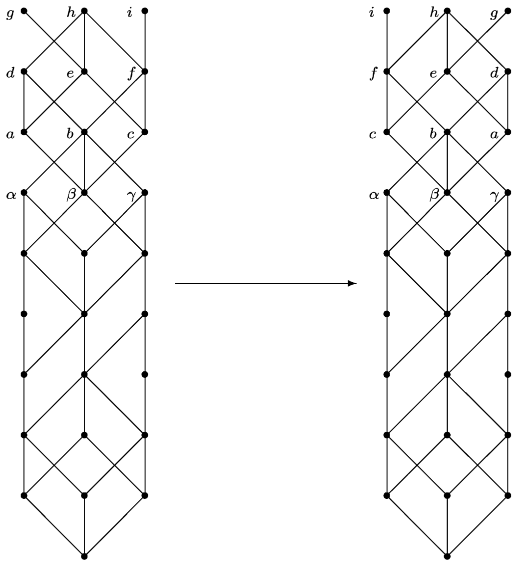


**Fig. 9.** Wherever a $seg_e$ occurs in a stack, we may flip those elements that occur above it, exchanging right for left, while maintaining the order relation.

switching the left-hand elements with the right, while maintaining the ordering of these elements. This is illustrated in Fig. 9.

In the case where $P$ is not a switch stack, there is a twist that occurs above any $seg_e$ in the stack. It shall be shown in the following lemma that this forces an orientation on the portion of the stack which occurs above the twist.

**Lemma 4.3.** *Let $P$ be a stack of rank $n$, and let $f$ be an automorphism on $P$. If $P$ is not a switch stack, $f(a) = a$ for all $a$ of rank $n-1$.*

**Proof.** If $P$ is not a switch stack, then $P$ has at least one twist. Suppose that $P$ has a twist of rank $r$, and that there is no twist having rank greater than $r$ (which means there is no $seg_e$ of rank $r$ or greater). By Lemma 4.2, the elements of rank $r$ are fixed under $f$. If $r = n-1$ then we are done. The case $r = n$ is easy. Suppose now $r < n-1$. If there is no $seg_a$ or $seg_e$ of rank $i$ or higher, and $f(x) = x$ for all $x$ of rank $i$, then $f(y) = y$ for all $y$ of rank $i+1$. □

As alluded to earlier, we wish to determine the number of distinct stacks one can construct by concatenating a given segment onto some base stack. We begin by considering the case where the base stack is not a switch stack, and has a $seg_b$ as its uppermost segment.

**Lemma 4.4.** *Let $P$ be a stack of rank $n$ such that the segment of rank $n-1$ is isomorphic to $seg_b$, and let $S$ be a segment isomorphic to either $seg_d$ or $seg_e$. Also, let $\phi_1, \phi_2, \phi_3$ be concatenation functions from $P$ to $S$ corresponding to the three arrangements given in Fig. 5. If $P$ is not a switch stack, then $P\&_{\phi_1}S$, $P\&_{\phi_2}S$, and $P\&_{\phi_3}S$ are pairwise non-isomorphic.*

**Proof.** Let $x, y, z \in P$ have rank $n-1$, and let $k$ be the minimal key element of $S$. Any isomorphism $f : P\&_{\phi_a}S \to P\&_{\phi_b}S$ $(1 \leqslant a < b \leqslant 3)$ must act as an automorphism on $P$ and $S$. Hence, $f$ is fixed pointwise on $\{x, y, z\}$ in $P$ (by Lemma 4.3), and $f(k) = k$ in $S$. Now, for each $\phi_i$ there is a single element of $\{x, y, z\}$ which is not a lower cover of $k$ in $P\&_{\phi_i}S$; we shall say $x \not\prec k$ in $P\&_{\phi_a}S$ and $y \not\prec k$ in $P\&_{\phi_b}S$. But then we have $y \leqslant k$ in $P\&_{\phi_a}S$, and $f(y) \nleqslant f(k)$ in $P\&_{\phi_b}S$, which implies that $f$ is not an isomorphism. □

In terms of Fig. 5, the preceding lemma says that if $P$ is not a switch stack, adding each of the three different arrangements of $seg_d$ or $seg_e$ to $P$ yields distinct stacks. In the case where $P$ is a switch stack, our earlier discussion showed that the two asymmetrical arrangements of the segments are in fact equivalent, yielding isomorphic stacks. We shall now demonstrate that, except in a particular circumstance, the asymmetrical arrangements of these segments are not equivalent to the symmetrical arrangement.

**Lemma 4.5.** *Let $P$ be a switch stack of rank $n$ and assume $P$ has a key element of rank $n-1$. Let $P\&_{\phi_1}S$ and $P\&_{\phi_2}S$ be stacks of rank $n+1$ such that the segment of rank $n-1$ is isomorphic to $seg_b$ and $S$ is isomorphic to either $seg_d$ or $seg_e$. Suppose that, from Fig. 5, $\phi_1$ comes from gluing $S$ onto $P$ asymmetrically, and $\phi_2$ symmetrically. Then $P\&_{\phi_1}S \ncong P\&_{\phi_2}S$.*

**Proof.** Then $P\&_{\phi_1}S$ has a twist of rank $n$ but $P\&_{\phi_2}S$ does not. □

We now consider the case where the stack is not a switch stack, and the uppermost segment is a $seg_d$.

**Lemma 4.6.** *Let $P$ be a stack of rank $n$ such that the segment of rank $n-1$ is isomorphic to $seg_d$, and let $S$ be a segment isomorphic to $seg_f$. Also, let $\phi_1$, $\phi_2$ be concatenation functions from $P$ to $S$ corresponding to the two arrangements given in Fig. 5. If $P$ is not a switch stack, then $P\&_{\phi_1}S \ncong P\&_{\phi_2}S$.*

**Proof.** Define $x, y, z \in P$, and $k \in S$ as in the proof of Lemma 4.4. For each $\phi_i$, there is a single element of $\{x, y, z\}$ which is covered by $k$ in $P\&_{\phi_i}S$; say $x \prec k$ in $P\&_{\phi_1}S$, and $y \prec k$ in $P\&_{\phi_2}S$. Since we must have $f(x) \prec f(k)$ in $P\&_{\phi_2}S$, we conclude that $f(x) = y$ in $P\&_{\phi_2}S$, contradicting Lemma 4.3. □

As before, if $P$ is not a switch stack, each of the arrangements of $seg_f$ in Fig. 5 yields a distinct stack. Also, we can see that if $P$ is a switch stack, then the two arrangements are equivalent.

Since only $seg_d$, $seg_e$ and $seg_f$ have different arrangements in Fig. 5, we have only one case left to consider: the case where a stack has a $seg_f$ as its uppermost segment. (Note that no stack of this form may be a switch stack.)

**Lemma 4.7.** *Let $P$ be a stack of rank $n$ such that the segment of rank $n-1$ is isomorphic to $seg_f$, and let $S$ be a segment isomorphic to $seg_f$. If $\phi_1$, $\phi_2$ are concatenation functions from $P$ to $S$ corresponding to the two arrangements given in Fig. 5, then $P\&_{\phi_1}S \ncong P\&_{\phi_2}S$.*

**Proof.** Obvious. □

We must now move from stacks to 3-lattices, using the above rules to formulate a system of recurrence relations that give the number of distinct 3-lattices of a given rank. In this endeavor, it is convenient to give names to the following types of stack.

**Definition.** If a stack has a least element, but is not complete, then we say it is a *subcomplete stack*. If a subcomplete stack $S$ has rank $n$, and the segment of rank $n-1$ in $S$ is isomorphic to $seg_d$, then we say it is *d-subcomplete*. We define *e-subcomplete*, and *f-subcomplete* stacks analogously.

By checking Fig. 5, we see that we may "complete" an $e$-subcomplete stack by concatenating onto it a $seg_b$, and then a $seg_c$. Similarly, we may complete a $d$- or $f$-subcomplete stack by concatenating a $seg_g$ onto it, and proceeding as before. Clearly, any such completion is unique up to isomorphism, so that there is a one-to-one correspondence between the number of $d$-, $e$- or $f$-subcomplete stacks and their completions. This is important because, by again checking Fig. 5, we can see that every 3-lattice of rank $> 3$ is a completion of some $d$-, $e$- or $f$-subcomplete stack.

Let $L_n$ be the number of distinct 3-lattices of rank $n$, and let $S_n$ be the number of rank $n$ switch stacks which are also 3-lattices (we shall refer to such 3-lattices as *switch lattices*). To obtain an expression for $L_n$, we first find one for $S_n$. In the following arguments, we shall often assume that $n > 8$, since there are special cases that apply when $n$ is small; but we will explicitly mention the values of $n$ for which our results are valid.

Clearly, if $L$ is a 3-lattice, then it is a switch lattice if and only if $L - \{1\}$ is a switch stack. (Moreover, a $d$-subcomplete stack $P$ is a switch stack if and only if its completion is a switch lattice.) We proceed by determining how we may concatenate segments onto $L - \{1\}$ so as to obtain a new switch stack. Since no switch stack may have a $seg_f$ as its top segment, we may conclude that every switch lattice $L$ of rank $> 3$ is a completion of a $d$- or $e$-subcomplete stack. By Fig. 5, if $S$ is a segment isomorphic to $seg_d$, then $(L - \{1\})\&_\phi S$ is a switch stack if and only if $\phi$ does not induce a twist (if and only if we use the symmetric $seg_d$). Also, if $L'$ is any 3-lattice, and $S$ is isomorphic to $seg_e$, we see that $(L' - \{1\})\&_\phi S$ is again a switch stack.

Thus, there are $S_n$ $d$-subcomplete switch stacks of rank $n$, where $n \geqslant 3$, since for $n > 3$ there is only a single way to concatenate a $seg_d$ onto a given $L - \{1\}$ so that there is no twist. Similarly, there are $3(L_n - S_n) + 2(S_n) = 3L_n - S_n$ $e$-subcomplete stacks of rank $n$, where $n > 3$, since there are three distinct ways to concatenate a $seg_e$ onto a non-switch stack, and two ways to do so with a switch stack. As noted previously, there is a unique completion of each $d$- or $e$-subcomplete stack. The completion of a $d$-subcomplete stack has rank three greater than the original stack, while the completion of an $e$-subcomplete stack has rank two greater, so we have

$$S_n = 3L_{n-2} - S_{n-2} + S_{n-3} \quad \text{for } n > 5. \tag{1}$$

We may now consider $L_n$. Let $d_n$, $e_n$ and $f_n$ be the number of rank $n$ $d$-, $e$- and $f$-subcomplete stacks, respectively. As we noted above, $L_n = d_{n-3} + e_{n-2} + f_{n-3}$ for $n > 3$. By arguments similar to those above, $d_n = e_n = 3L_n - S_n$ for $n > 3$. To find an expression for $f_n$, we note that for each $d$-subcomplete stack, we may concatenate a $seg_f$ onto it in two distinct ways if it is not a switch stack, and one if it is. Also, for each $f$-subcomplete stack, we may always concatenate a $seg_f$ onto it in two distinct ways. Thus, $f_n = 2d_{n-1} - S_{n-1} + 2f_{n-1}$ for $n \geqslant 4$ (recalling that there are $S_n$ $d$-subcomplete switch stacks of rank $n$ for $n \geqslant 3$). Substituting $f_{n-1} = L_{n+2} - d_{n-1} - e_n$ into the previous equation for $n > 1$, we have $f_n = 2d_{n-1} - S_{n-1} + 2(L_{n+2} - d_{n-1} - e_n)$, or $f_n = 2L_{n+2} - S_{n-1} - 6L_n + 2S_n$ for $n \geqslant 4$. Therefore,

$$\begin{aligned} L_n &= 3L_{n-3} - S_{n-3} + 3L_{n-2} - S_{n-2} + 2L_{n-1} - S_{n-4} - 6L_{n-3} + 2S_{n-3} \\ &= 2L_{n-1} + 3L_{n-2} - S_{n-2} - 3L_{n-3} + S_{n-3} - S_{n-4} \quad \text{for } n > 6. \end{aligned}$$

By substituting $S_{n-1} = 3L_{n-3} - S_{n-3} + S_{n-4}$ for $n > 6$, we have

$$L_n = 2L_{n-1} + 3L_{n-2} - S_{n-2} - S_{n-1} \quad \text{for } n > 6. \tag{2}$$

We now find an expression for $S_n$ which is solely a function of its own terms, which will allow us to determine the generating function of $L_n$.

Rearranging (1), and adjusting coefficients, we find $L_n = (S_{n+2} + S_n - S_{n-1})/3$ for $n > 3$. Substituting this back into (2), we have

$$\begin{aligned} L_n &= (2/3)(S_{n+1} + S_{n-1} - S_{n-2}) + (S_n + S_{n-2} - S_{n-3}) - S_{n-2} - S_{n-1} \\ &= (2/3)(S_{n+1} + S_{n-1} - S_{n-2}) + S_n - S_{n-1} - S_{n-3} \quad \text{for } n > 6. \end{aligned}$$

Substituting into (1):

$$\begin{aligned} S_n &= 3\big[(2/3)(S_{n-1} + S_{n-3} - S_{n-4}) + S_{n-2} - S_{n-3} - S_{n-5})\big] - S_{n-2} + S_{n-3} \\ &= 2S_{n-1} + 2S_{n-2} - 2S_{n-4} - 3S_{n-5} \quad \text{for } n > 8. \end{aligned} \tag{3}$$

We may now formulate the generating function for $S_n$. Let

$$g(x) = S_0 + S_1 x + S_2 x^2 + \cdots.$$

Then

$$\begin{aligned}&g(x) - 2g(x)x - 2g(x)x^2 + 2g(x)x^4 + 3g(x)x^5\\&\quad = S_0 + (S_1 - 2S_0)x + (S_2 - 2S_1 - 2S_0)x^2 + (S_3 - 2S_2 - 2S_1)x^3\\&\qquad + (S_4 - 2S_3 - 2S_2 + 2S_0)x^4 + (S_5 - 2S_4 - 2S_3 + 2S_1 + 3S_0)x^5\\&\qquad + (S_6 - 2S_5 - 2S_4 + 2S_2 + 3S_1)x^6 + \cdots. \qquad (4)\end{aligned}$$

By examining Fig. 5 and Figs. A.2–A.5, we see that $S_0 = S_1 = S_2 = S_4 = 0$, and $S_3 = S_5 = 1$. Also $S_6 = 1$, $S_7 = 2$, and $S_8 = 3$. (For $n = 7$, $abd_0f_0gbc$ is not a switch lattice, but both of the words $abe_0be_0bc$ and $abe_0be_1bc$ are switch lattices. For $n = 8$, both of the words $abd_0gbe_0bc$ and $abd_0gbe_1bc$ are switch lattices, neither $abd_0f_0f_0gbc$ nor $abd_0f_0f_1gbc$ is a switch lattice, and $abe_0bd_0gbc$ is a switch lattice while $abe_0bd_1gbc$ is not.) When $n > 6$, (3) holds (check directly for $n = 7, 8$), and so the coefficients of the above polynomial are 0 for all $n > 6$. Substituting these values into (4), we have

$$g(x) - 2g(x)x - 2g(x)x^2 + 2g(x)x^4 + 3g(x)x^5 = x^3 - 2x^4 - x^5 - x^6.$$

Solving for $g(x)$:

$$g(x) = \frac{x^3 - 2x^4 - x^5 - x^6}{1 - 2x - 2x^2 + 2x^4 + 3x^5}. \qquad (5)$$

Now, let

$$f(x) = L_0 + L_1x + L_2x^2 + \cdots$$

so that

$$\begin{aligned}&f(x) - 2f(x)x - 3f(x)x^2 + g(x)x + g(x)x^2\\&\quad = L_0 + (L_1 - 2L_0 + S_0)x + (L_2 - 2L_1 - 3L_0 + S_1 + S_0)x^2\\&\qquad + (L_3 - 2L_2 - 3L_1 + S_2 + S_1)x^3 + (L_4 - 2L_3 - 3L_2 + S_3 + S_2)x^4\\&\qquad + (L_5 - 2L_4 - 3L_3 + S_4 + S_3)x^5 + (L_6 - 2L_5 - 3L_4 + S_5 + S_4)x^6 + \cdots. \qquad (6)\end{aligned}$$

As before, we check Fig. 5 for the initial values, finding that $L_n = S_n$ for $n < 7$. Substituting these values into (6) we see that

$$f(x) - 2f(x)x - 3f(x)x^2 + g(x)x + g(x)x^2 = x^3 - x^4 - x^5.$$

Solving for $f(x)$:

$$f(x) = \frac{x^3 - x^4 - x^5 - g(x)x - g(x)x^2}{1 - 2x - 3x^2}. \qquad (7)$$

We thus get the solution to the problem from Stanley's 1986 text *Enumerative Combinatorics* by using a supercomputer to analyze the posets of cardinality $n \leqslant 2$.

**Theorem 4.8.** *Let $f_3(n)$ be the number of posets $P$ of cardinality $n$ such that, for $0 < i < n$, $P$ has exactly 3 down-sets of cardinality $i$. Then*

$$\sum_{n=0}^{\infty} f_3(n)x^n = \frac{1 - 3x - 5x^2 + 10x^3 + 14x^4 + 7x^5 - 6x^6 - 15x^7 - 7x^8 - 5x^9 - 3x^{10}}{1 - 4x - x^2 + 10x^3 + 8x^4 - x^5 - 12x^6 - 9x^7}. \qquad (8)$$

No attempt has been made to factor the numerator or denominator, such as the factor of $1 + x$.

As was stated in Section 1, in *Enumerative Combinatorics* Stanley first asks to count the posets enumerated by $f_3(n)$ such that the only three-element antichains are the set of minimal elements and the set of maximal elements, and then asks to use the fact that this number is $2^{n-7}$ for $n \geqslant 7$ to find $\sum f_3(n)x^n$. As the referee points out, for $n \geqslant 7$ this class of posets corresponds to the 3-lattices *abdff…fgbc*. (This is because the only element $< 1$ in the corresponding 3-lattice with 3 lower covers must be the join of the atoms, and dually.) We can fix the orientation of the first $d$ and $f$ segments, and thus there are $2^{n-7}$ ways to orient the other $f$ segments.

## 5. Conclusion: A question of B. Davey from the 1981 Banff Conference on Ordered Sets

The answer to the question about Fig. A.6 can be determined by computing the least upper bound of two of the atoms.

At the 1981 Banff Conference on Ordered Sets, B. Davey asked "a general, waffling-type question."

He said, "The duality between finite distributive lattices and finite ordered sets has often been used to answer algebraic questions concerning lattices via considerations of ordered sets." He then posed the

**Problem.** (B. Davey, 1981, see [5, p. 847].) Give a single example of a question concerning ordered sets which can be settled by transferring it to a question about distributive lattices and then using algebraic techniques.

It is not clear to us how one goes from the *posets* of cardinality $n$ with Stanley's property directly to those of cardinality $n+1$. (See Figs. A.2–A.5.) Indeed, two 3-lattices whose words differ by one letter (or the orientation of one segment) can correspond to posets whose structures are very different, and not so clearly related. It also seems difficult to prove by hand facts like $f_3(8) = 6$ without the flow diagram of Fig. 5. Hence, we regard our solution to Stanley's problem also as a "solution" to Davey's "problem" from the 1981 Banff Conference on Ordered Sets. Of course, the calculation of $f_2(n)$, an exercise in *Enumerative Combinatorics,* also involves looking at "2-lattices," but no results about lattices much deeper than Birkhoff's theorem are used.

[The answer to the question about Fig. A.7 is: You are correct (see [3, Proposition 4.1]).]

## Acknowledgments

We would like to thank Professor Michael Roddy of Brandon University; we would also like to thank the referee for her or his comments.

## Appendix A

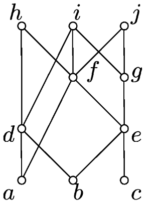


**Fig. A.1(i).** A poset with 3 order ideals of each cardinality $0 < i < 10$.

**Table A.1**
Order ideals of cardinality $i$ in the poset of Fig. A.1.

| $i$ | |
|---|---|
| 0 | $\emptyset$ |
| 1 | $a, b, c$ |
| 2 | $ab, ac, bc$ |
| 3 | $abc, abd, bce$ |
| 4 | $abcd, abce, bceg$ |
| 5 | $abcde, abcef, abceg$ |
| 6 | $abcdef, abcdeg, abcefg$ |
| 7 | $abcdefg, abcdefh, abcefgj$ |
| 8 | $abcdefgh, abcdefgi, abcdefgj$ |
| 9 | $abcdefghi, abcdefghj, abcdefgij$ |
| 10 | $abcdefghij$ |

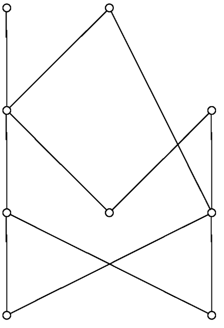

**Fig. A.1(ii).** Another poset with 3 order ideals of each cardinality $0 < i < 9$.

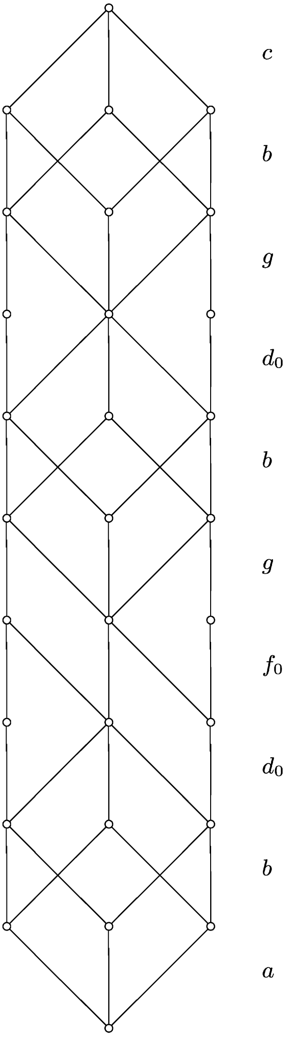


**Fig. A.1(iii).** The distributive lattice corresponding to Fig. A.1(i) (word $abd_0f_0gbd_0gbc$).

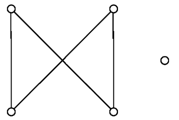

**Fig. A.2(i).** The poset illustrating $f_3(5) = 1$.

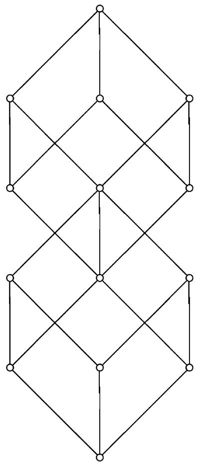

**Fig. A.2(ii).** The corresponding distributive lattice (word $abe_0bc$).

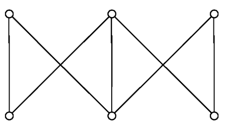

**Fig. A.3(i).** The poset illustrating $f_3(6) = 1$.

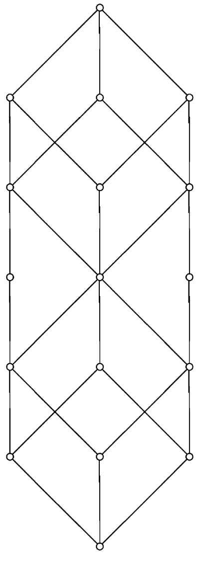

**Fig. A.3(ii).** The corresponding distributive lattice (word $abd_0gbc$).

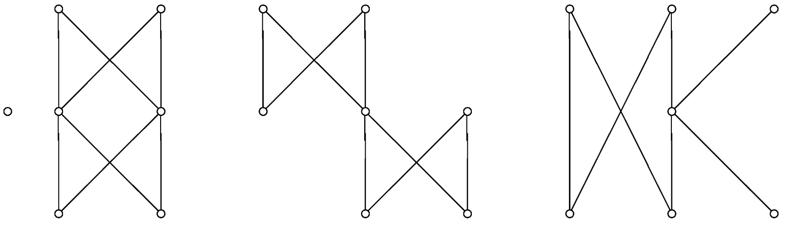

**Fig. A.4(i).** The posets illustrating $f_3(7) = 3$.

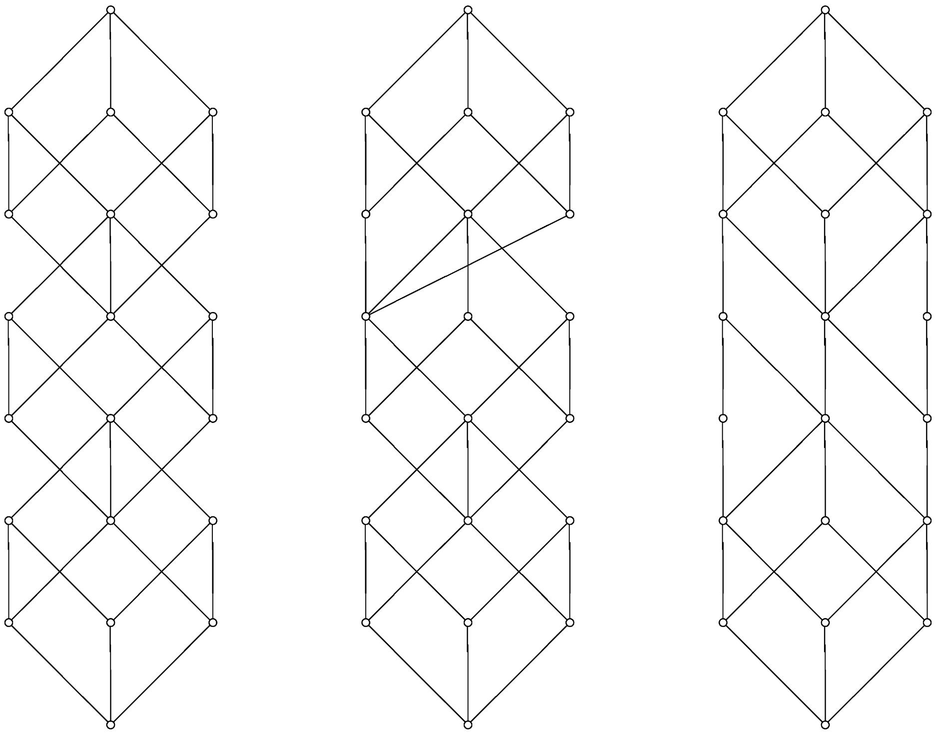

**Fig. A.4(ii).** The corresponding distributive lattices (words $abe_0be_0bc$, $abe_0be_1bc$, and $abd_0f_0gbc$).

**Fig. A.5(i).** Two posets that, together with their duals, contribute to $f_3(8) = 6$.

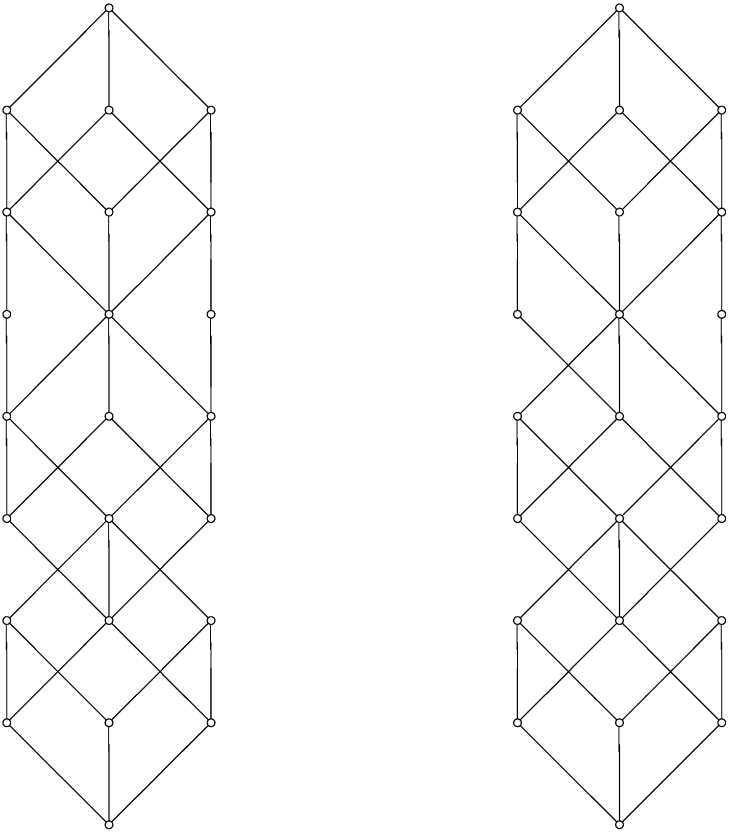

**Fig. A.5(ii).** The corresponding distributive lattices (words $abe_0bd_0gbc$ and $abe_0bd_1gbc$).

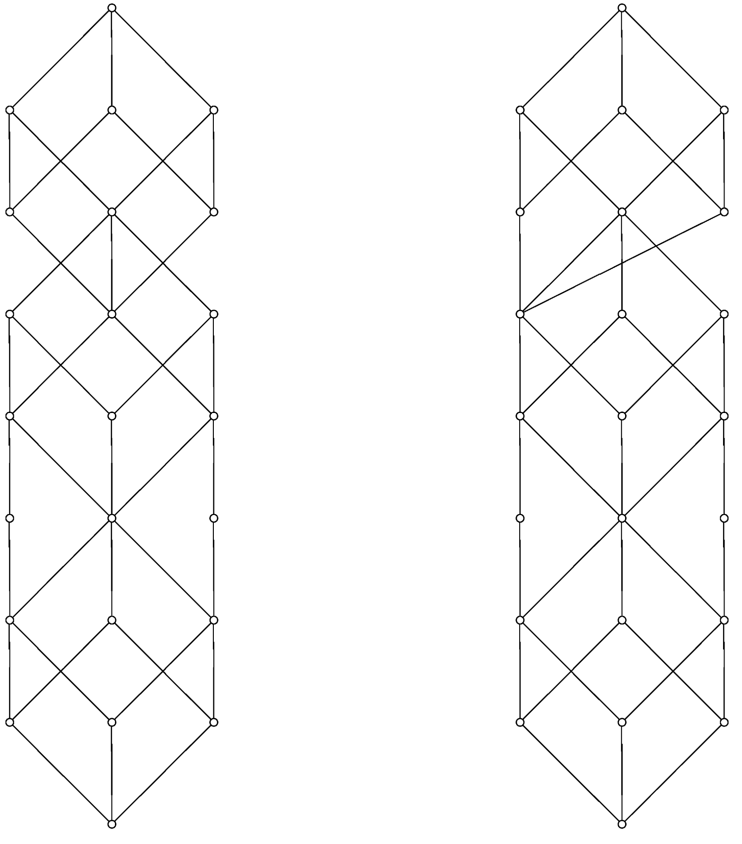

**Fig. A.5(iii).** Their duals (words $abd_0gbe_0bc$ and $abd_0gbe_1bc$).

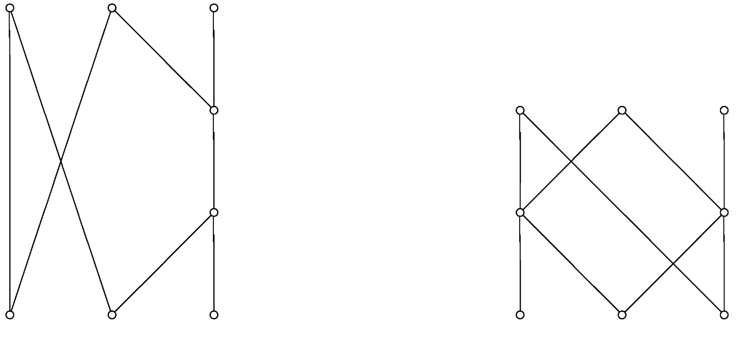

**Fig. A.5(iv).** Two of the (self-dual) posets that contribute to $f_3(8) = 6$.

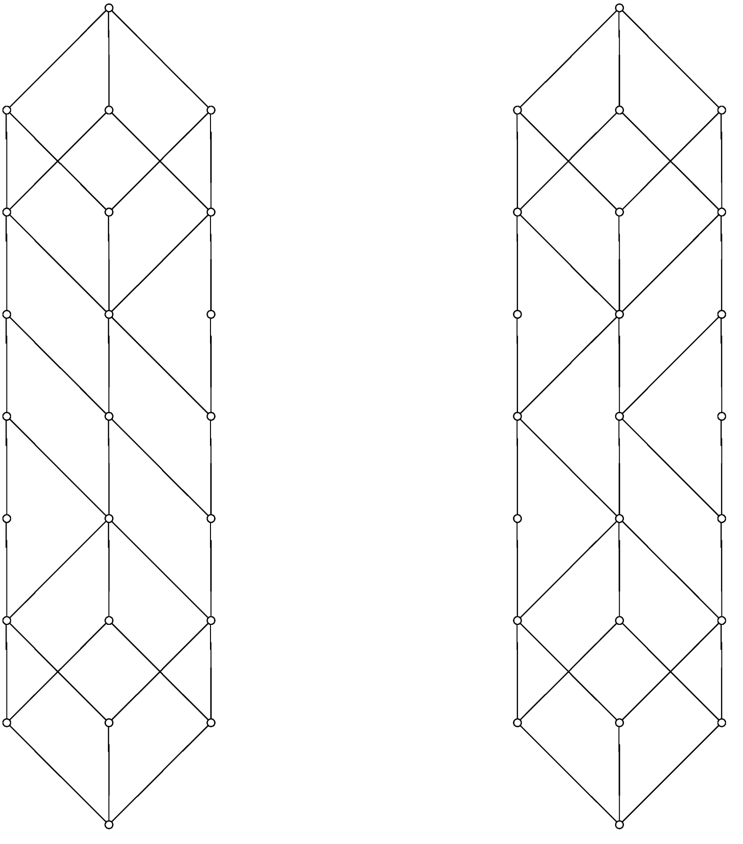

**Fig. A.5(v).** The corresponding distributive lattices (words $abd_0 f_0 f_0 gbc$ and $abd_0 f_0 f_1 gbc$).

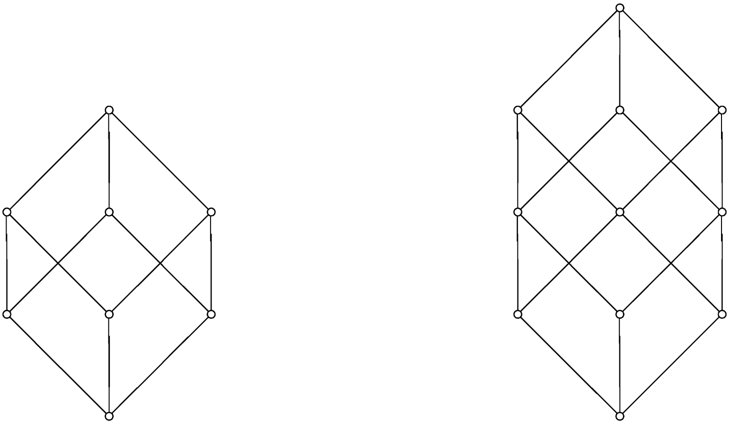

**Fig. A.6.** Is the poset on the right a (distributive) lattice?

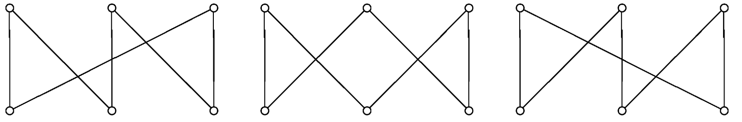

**Fig. A.7(i).** Three drawings of a 6-crown.

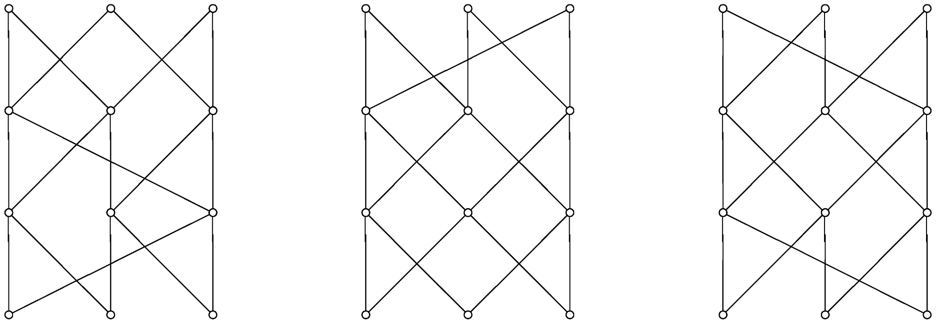

**Fig. A.7(ii).** Three ways to “stack” a 6-crown; which two are isomorphic?

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