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Quasi-Differential Posets and Cover Functions of Distributive Lattices II: A Problem in Stanley's *Enumerative Combinatorics*

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Abstract. A distributive lattice L with 0 is *finitary* if every interval is finite. A function $f : \mathbb{N}_0 \to \mathbb{N}_0$ is a *cover function* for L if every element with n lower covers has $f(n)$ upper covers. All non-decreasing cover functions have been characterized by the author ([2]), settling a 1975 conjecture of Richard P. Stanley. In this paper, all finitary distributive lattices with cover functions are characterized. A problem in Stanley's *Enumerative Combinatorics* is thus solved.

Key words. Differential poset, Fibonacci lattice, Distributive lattice, (partially) Ordered set, Cover function

A. Preliminaries

1. The Problem

In this paper, we continue the investigations begun by Stanley in [3], in which he studies certain distributive lattices related to the Fibonacci numbers.

Many of these lattices have the following property: whenever two elements have the same number (n) of immediate predecessors, then they have the same number ($f(n)$) of immediate successors. Hence one may define a *cover function* $f : \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Problem (Stanley, [4], [6], p. 157). "*Can all cover functions f(n) be explicitly characterized?*"

We answer this question by characterizing all cover functions and their corresponding lattices (Theorem 11.1).

In the rest of Part A we shall define our terms (§2) and state the problem precisely (§3). Then we shall present background material more directly related to

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the problem and give some basic examples. We repeat much of the introductory material from [2].

In Part B we shall solve the problem by doing a case-by-case analysis of all possible cover functions. (Because of our previous work, we need only consider non-non-decreasing cover functions.)

2. *General Definitions, Notation, and Basic Theory*

For basic facts and notation, see [1] or [6].

Let P be a poset. We denote the least element by 0_P or 0 if it exists.

Let $p, q \in P$. We say p is a *lower cover* of q and q is an *upper cover* of p (denoted $p \lessdot q$) if $p < q$ and there is no $r \in P$ such that $p < r < q$. We denote the set of lower covers of p by $\mathrm{LC}(p)$. An element is *(join-) irreducible* if it has a unique lower cover. Let $\mathrm{Irr}(P)$ denote the poset of irreducibles of P.

A subset $Q \subseteq P$ is a *down-set* (or *order ideal*) if $p \in P$, $q \in Q$, and $p \leqslant q$ imply $p \in Q$ (Fig. 1).

The family of *finite* down-sets of P is denoted $\mathcal{O}_f(P)$. For $R \subseteq P$,

$$\downarrow R = \{p \in P \mid p \leqslant r \text{ for some } r \in R\};$$

if R is a singleton $\{r\}$, we simply write $\downarrow r$, and $\overset{\circ}{\downarrow} r$ denotes $(\downarrow r)\backslash\{r\}$. (Note that $\downarrow R$ is a down-set.)

Let P and Q be posets. The *disjoint sum* of P and Q, $P + Q$, is the poset with underlying set $P \cup Q$ such that p and q are incomparable for all $p \in P$ and $q \in Q$ (Fig. 2). The *ordinal sum* of P and Q, $P \oplus Q$, is the poset on $P \cup Q$ such that $p < q$ for all $p \in P$ and $q \in Q$ (Fig. 3).

If P has a greatest element and Q a least element, the *coalesced ordinal sum*, $P \boxplus Q$, is the poset obtained by identifying these two elements (Fig. 4).

The *direct product* $P \times Q$ is the set of pairs (p, q) ordered coordinate-wise: $(p, q) \leqslant (p', q')$ if $p \leqslant p'$ and $q \leqslant q' (p, p' \in P, q, q' \in Q)$ – see Figs. 5a and b.

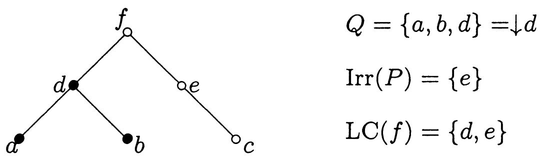

Fig. 1. A down-set Q of P

Fig. 2. The disjoint sum

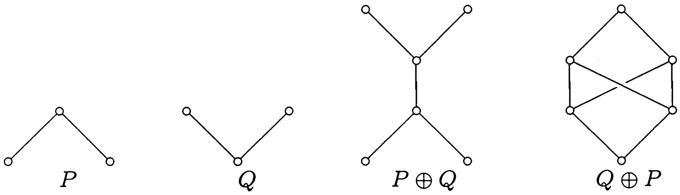

Fig. 3. The ordinal sum

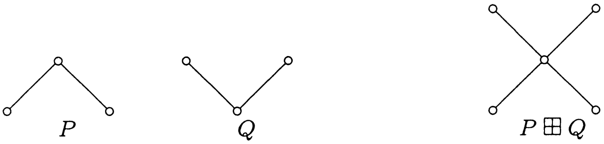

Fig. 4. The coalesced ordinal sum

An *antichain* is a poset in which distinct elements are incomparable; a *chain* is a totally ordered set. For $n \in \mathbb{N}_0$, the n-element antichain is denoted $\bar{n}$ and the n-element chain is denoted $\mathbf{n}$ (Fig. 6).

A lattice L is *finitary* if it has a 0 and $\downarrow a$ is finite for all $a \in L$. It is well known that a finitary distributive lattice may be identified with $\mathcal{O}_{\mathrm{f}}(P)$ where $P = \mathrm{Irr}L$ ([6], 3.4.3).

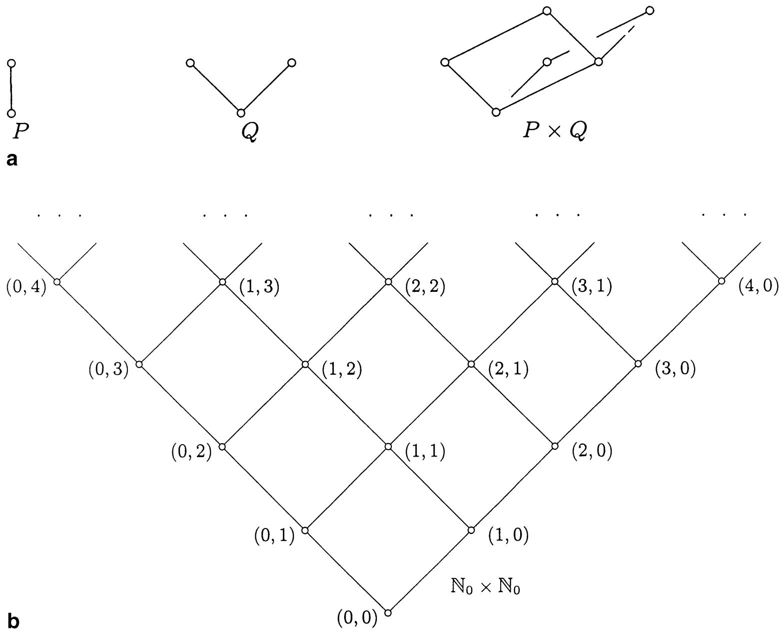

Fig. 5. Direct products

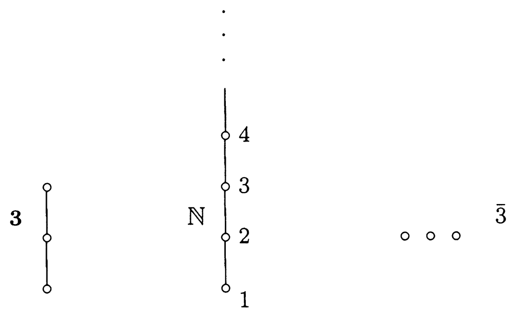

Fig. 6. Chains and an antichain

If we do simply say that $L = \mathcal{O}_{\mathrm{f}}(P)$, then $I \lessdot J$ in L if and only if $I = J \backslash \{j\}$ for a maximal element $j \in J$ (now viewed as a subposet of P).

For posets P and Q, $\mathcal{O}_{\mathrm{f}}(P + Q) \cong \mathcal{O}_{\mathrm{f}}(P) \times \mathcal{O}_{\mathrm{f}}(Q)$, and, if P is finite, $\mathcal{O}_{\mathrm{f}}(P \oplus Q) \cong \mathcal{O}_{\mathrm{f}}(P) \boxplus \mathcal{O}_{\mathrm{f}}(Q)$. In particular, $\mathcal{O}_{\mathrm{f}}(\mathbf{1} \oplus Q) \cong \mathbf{1} \oplus \mathcal{O}_{\mathrm{f}}(Q)$. (See Figs. 7 and 8 and [1], Chapter 5.)

Let Y denote *Young's lattice* (a lattice of great interest to combinatorialists). It is the poset of sequences $(a_1, a_2, \ldots) \in \mathbb{N}_0^{\omega}$ with finitely many non-zero coordinates such that $a_1 \geqslant a_2 \geqslant \cdots$. We will identify Young's lattice with $\mathcal{O}_{\mathrm{f}}(\mathbb{N}_0 \times \mathbb{N}_0)$ (Fig. 9).

3. Definition of Cover Functions and Known Results

Let L be a finitary distributive lattice. A function $f : \mathbb{N}_0 \to \mathbb{N}_0$ is a *cover function* for L if every element with (exactly) n lower covers has (exactly) $f(n)$ upper covers.

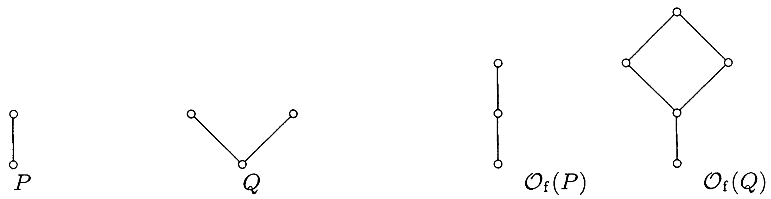

Fig. 7. The lattice of down-sets

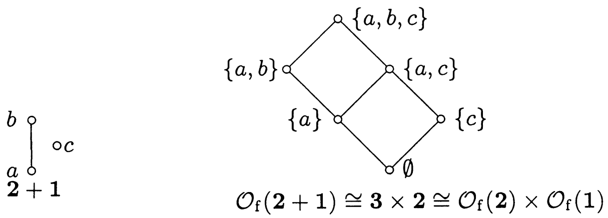

Fig. 8. The lattice of down-sets

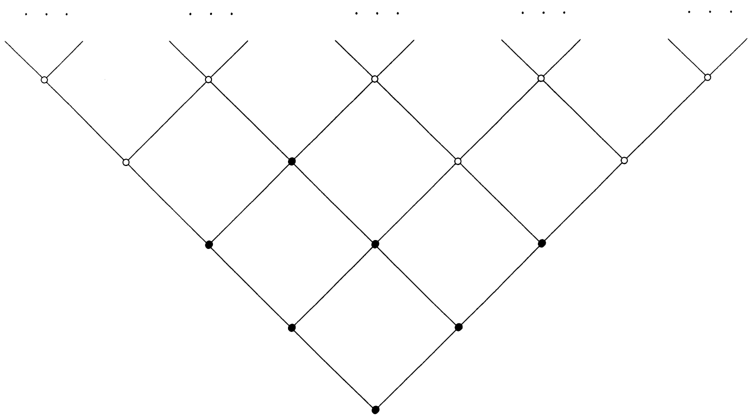

Fig. 9. An *element* of Young's lattice

(The definition comes from [3], §3 and [6], p. 157; cf. the definition of *differential posets* in [5].)

The first three examples come from [3].

Example 3.1. For $k \in \mathbb{N}$, the constant function $f(n) = k$ $(n \in \mathbb{N}_0)$ is a cover function for $\mathbb{N}_0^k$ (Figs. 10a and b).

[We note that $f(n)$ could take any value for $n > k$.]

Example 3.2. For $k \in \mathbb{N}$, the function $f(n) = k + n$ $(n \in \mathbb{N}_0)$ is a cover function for Y^k.

Example 3.3. For $k \in \mathbb{N}_0$, any function $f : \mathbb{N}_0 \to \mathbb{N}_0$ with $f(n) = k - n$ $(0 \leqslant n \leqslant k)$ is a cover function for $\mathbf{2}^k$ (Fig. 11).

In fact, we have:

Proposition 3.4 ([3], §3, Proposition 2). *If L is a finite distributive lattice with a cover function, then $L \cong \mathbf{2}^r$ for some $r \in \mathbb{N}_0$.* □

We have constructed the following examples:

Example 3.5. For $k \geqslant 2$, the function

$$f(n) = \begin{cases} k - n & \text{if } 0 \leqslant n < k, \\ k & \text{if } n = k, \\ * & \text{otherwise,} \end{cases}$$

where $n \in \mathbb{N}_0$, is a cover function for $\boxplus_{i=1}^{\infty} \mathbf{2}^k$ (Fig. 12).

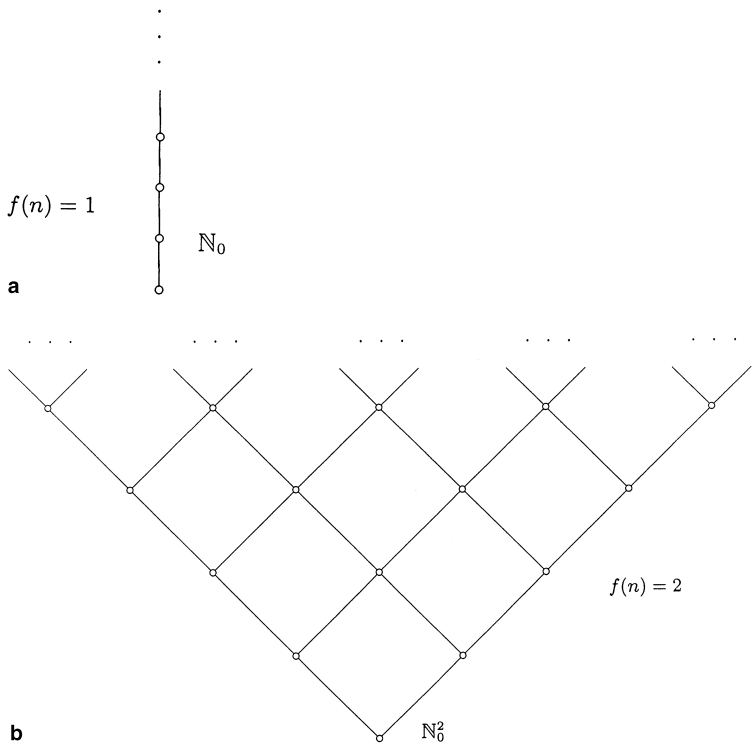

Fig. 10. Cover functions

Example 3.6. For $k \geqslant 2$, the function

$$f(n) = \begin{cases} 1 & \text{if } n = 0, \\ k & \text{if } 1 \leqslant n \leqslant k, \\ * & \text{otherwise,} \end{cases}$$

where $n \in \mathbb{N}_0$, is a cover function for $\mathbf{1} \oplus \mathbb{N}_0^k$ (Fig. 13).

Example 3.7. The poset $L = Y \backslash \{0_Y\}$ is still a finitary distributive lattice [with $\mathrm{Irr}(L) \cong (\mathbb{N}_0 \times \mathbb{N}_0) \backslash \{(0,0)\}$], and it has cover function

$$f(n) = \begin{cases} 2 & \text{if } n = 0, \\ n+1 & \text{if } n \geqslant 1, \end{cases}$$

where $n \in \mathbb{N}_0$.

Example 3.8. Another "sporadic" example is the lattice $L = \mathbf{2}^2 \oplus \mathbb{N}$, which has cover function

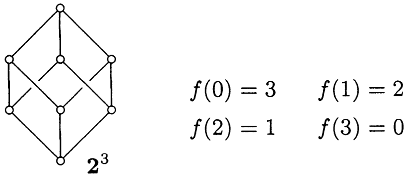

Fig. 11. A Cover function

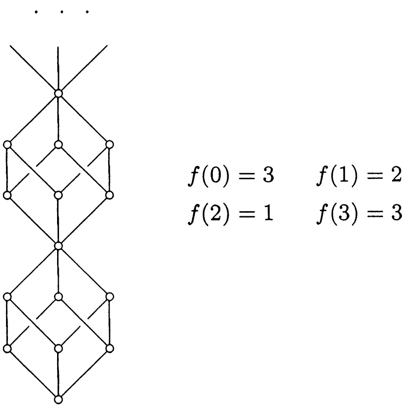

Fig. 12. A Cover function

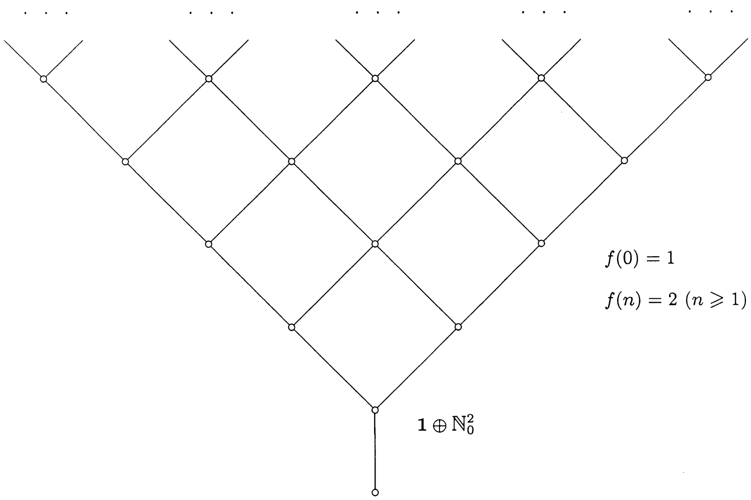

Fig. 13. Cover functions

$$f(n) = \begin{cases} 2 & \text{if } n = 0, \\ 1 & \text{if } 1 \leqslant n \leqslant 2, \\ * & \text{otherwise,} \end{cases}$$

where $n \in \mathbb{N}_0$ (Fig. 14).

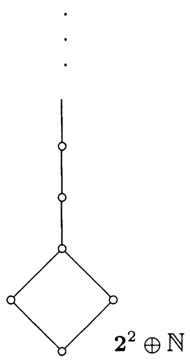

Fig. 14. A cover function

Even though we have seen that a given lattice L may have more than one cover function, any two lattices with the same cover function must be isomorphic:

Proposition 3.9 ([3], §3; [6], pp.157, 180). *There is at most one finitary distributive lattice with a given cover function* (*up to isomorphism*). □

In [3], Stanley conjectures that the only *non-decreasing* cover functions are the constant functions and functions of the form $f(n) = n + k$ for some constant k. (He proves that no cover function has the form $f(n) = an + k$ if $|a| \geqslant 2$.) We settle the conjecture in [2].

In Part B we prove the following (Theorem 11.1):

Theorem. *If L is a finitary distributive lattice with a cover function, then one of the following holds:*

(1) $L \cong \mathbb{N}_0^k$ $(k \geqslant 1)$;
(2) $L \cong Y^k$ $(k \geqslant 1)$;
(3) $L \cong \mathbf{1} \oplus \mathbb{N}_0^k$ $(k \geqslant 2)$;
(4) $L \cong Y \backslash \{0_Y\}$;
(5) $L \cong \mathbf{2}^k$ $(k \geqslant 0)$;
(6) $L \cong \boxplus_{i=1}^{\infty} \mathbf{2}^k$ $(k \geqslant 2)$;
(7) $L \cong \mathbf{2}^2 \oplus \mathbb{N}$.

B. Solution to the Problem

In Part B, L will denote a finitary distributive lattice with cover function $f : \mathbb{N}_0 \to \mathbb{N}_0$. Let $P = \mathrm{Irr}(L)$ and let $x_1, \ldots, x_m$ be its set of minimal elements. (It is clear that $m = f(0)$.)

We identify L with $\mathcal{O}_{\mathrm{f}}(P)$.

4. Useful Lemmas

We will use the following key lemmas repeatedly; they mostly follow from the characterization of the cover relation in $\mathcal{O}_{\mathrm{f}}(P)$ given in §2.

Lemma 4.1. *Let $n \in \mathbb{N}_0$. An element $I \in \mathcal{O}_{\mathrm{f}}(P)$ has (exactly) n lower covers in L if and only if I has n maximal elements in P.* □

Lemma 4.2. *If $I = \downarrow A$ where $A \subseteq P$ is an n-element antichain, then there are (exactly) $f(n)$ elements $p \in P \backslash I$ such that $LC(p) \subseteq I$.* □

Lemma 4.3. *Let $q \in P$; let $A = \{p \in P \backslash \downarrow q \mid LC(p) \subseteq \overset{\circ}{\downarrow} q\}$, and let $B = \{p \in P \backslash \downarrow q \mid LC(p) \subseteq \downarrow q\}$.*

Then:

(1) $A \subseteq B$;
(2) *for all* $r \in P,\ r \in B \backslash A$ *if and only if* $r \in \mathrm{Irr}(P)$ *and* $q \lessdot r$;
(3) *the upper covers of* $\downarrow q$ *in* L *are exactly the sets* $\downarrow q \cup \{b\}$ *for* $b \in B$. □

Lemma 4.4. *If $f(0) \geqslant 2$, then $f(0) - 1 \leqslant f(1) \leqslant f(0) + 1$.*

Proof. This result is Lemma 4.7 of [2]. □

Proposition 4.5. *If $f(0) \leqslant f(1)$, then one of the following holds*:

(1) $L \cong \mathbb{N}_0^k$ $(k \geqslant 1)$;
(2) $L \cong Y^k$ $(k \geqslant 1)$;
(3) $L \cong \mathbf{1} \oplus \mathbb{N}_0^k$ $(k \geqslant 2)$;
(4) $L \cong Y \backslash \{0_Y\}$;
(5) $L \cong \mathbf{1}$.

Proof. A careful look at the statements proved in [2] yields the result. □

Lemma 4.6. *For all $n \in \mathbb{N}_0,\ f(n) \geqslant f(0) - n$.*

Proof. The statement is trivial if $n = 0$ or $n \geqslant f(0)$.

Suppose $1 \leqslant n < f(0)$ and let $I = \{x_1, \ldots, x_n\}$. Then I has, in L, at least the $f(0) - n$ upper covers $I \cup \{x_i\} (n < i \leqslant f(0))$. □

In the rest of §4 *and in* §§5–10, *we shall assume that $f(0) \geqslant 2$ and $f(1) = f(0) - 1$.*

Corollary 4.7. *No x_i has an irreducible upper cover $(1 \leqslant i \leqslant f(0))$.*

Proof. Use Lemma 4.3. □

Corollary 4.8. *Let $C = \{x_3, \ldots, x_{f(0)}\}$ and let $D = \{p \in P \backslash \{x_1, x_2\} \mid LC(p) \subseteq \{x_1, x_2\}\}$.*

Then:

(1) $C \subseteq D$;
(2) *for all* $r \in P,\ r \in D \backslash C$ *if and only if* $LC(r) = \{x_1, x_2\}$;
(3) *the upper covers of* $\{x_1, x_2\}$ *in* L *are exactly the sets* $\{x_1, x_2, d\}$ *for* $d \in D$. □

Lemma 4.9. *The inequality* $f(0)-2 \leqslant f(2) \leqslant f(0)$ *holds.*

Proof. The first inequality is Lemma 4.6. Suppose for a contradiction that $f(2) \geqslant f(0)+1$. Since $\{x_1, x_2\}$ has, in L, at least the $f(0)-2$ upper covers $\{x_1, x_2, x_i\}$ $(3 \leqslant i \leqslant f(0))$, by Corollary 4.8 there are at least 3 elements $z \in P$ such that $\mathrm{LC}(z) = \{x_1, x_2\}$, u, v, and w (Fig. 15).

So $\downarrow u$ has *more* than $f(1) = f(0)-1$ upper covers in L, namely, $\downarrow u \cup \downarrow v$, $\downarrow u \cup \downarrow w$, and $\downarrow u \cup \{x_i\}$ $(3 \leqslant i \leqslant f(0))$, a contradiction. □

5. $f(2) = f(0)$

Lemma 5.1. *Let C and D be as in Corollary* 4.8. *Then*:

(1) $D \backslash C$ *has exactly two elements*, y_{12} *and* y'_{12};
(2) *for* $q = y_{12}$ *in Lemma* 4.3, $A = B = C \cup \{y'_{12}\}$.

Proof. (1) By Corollary 4.8(3), $D \backslash C$ has exactly $f(2) - (f(0)-2) = 2$ elements.

(2) By Lemma 4.3(3), B has exactly $f(1) = f(0)-1$ elements; but $C \cup \{y'_{12}\} \subseteq A$ has $f(0)-1$ elements, so $A = B = C \cup \{y'_{12}\}$. □

Corollary 5.2. *We have* $f(0) = 2$.

Proof. Assume for a contradiction that $f(0) \geqslant 3$ (Fig. 16).

Then $\downarrow y_{12} \cup \{x_3\}$ has *more* than $f(2) = f(0)$ upper covers in L, namely, $\downarrow y_{12} \cup \downarrow y'_{12} \cup \{x_3\}$, $\downarrow y_{12} \cup \downarrow y_{13}$, $\downarrow y_{12} \cup \downarrow y'_{13}$, $\downarrow y_{12} \cup \downarrow y_{23}$, $\downarrow y_{12} \cup \downarrow y'_{23}$, and $\downarrow y_{12} \cup \{x_3, x_i\}$ $(4 \leqslant i \leqslant f(0))$, a contradiction. □

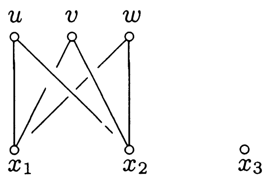

Fig. 15. An impossible scenario ($f(0) = 3$, $f(1) = 2$, $f(2) > 3$)

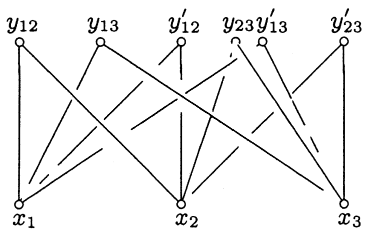

Fig. 16. An impossible scenario ($f(0) = 3$, $f(1) = 2$, $f(2) = 3$)

6. $f(2) = f(0) - 1$

Lemma 6.1. *Let C and D be as in Corollary* 4.8.

Then:

(1) *$D \backslash C$ has exactly one element*, y_{12};
(2) *y_{12} has a unique irreducible upper cover*, y'_{12}.

Proof. (1) By Corollary 4.8(3), $D \backslash C$ has exactly

$$f(2) - (f(0) - 2) = f(0) - 1 - (f(0) - 2) = 1$$

element.

(2) In L, $\downarrow y_{12}$ has at least the $f(0) - 2 = f(1) - 1$ upper covers $\downarrow y_{12} \cup \{x_i\}$ $(3 \leqslant i \leqslant f(0))$, so there is exactly one more element $y'_{12} \in P \backslash \downarrow y_{12}$ such that $LC(y'_{12}) \subseteq \downarrow y_{12}$. We have ruled out $\mathrm{LC}(y'_{12}) = \emptyset$, and Corollary 4.7 rules out $\mathrm{LC}(y'_{12}) = \{x_1\}$ or $\{x_2\}$; (1) and Corollary 4.8(2) rule out $\mathrm{LC}(y'_{12}) = \{x_1, x_2\}$. Hence $\mathrm{LC}(y'_{12}) = \{y_{12}\}$. □

Corollary 6.2. *We have* $f(0) = 2$.

Proof. Assume for a contradiction that $f(0) \geqslant 3$ and consider the elements of Lemma 6.1 (Fig. 17).

Then $\downarrow y_{12} \cup \{x_3\}$ has *more* than $f(2) = f(0) - 1$ upper covers in L, namely, $\downarrow y_{12} \cup \{x_3, x_i\}$ $(4 \leqslant i \leqslant f(0))$, $\downarrow y_{12} \cup \downarrow y_{13}$, $\downarrow y_{12} \cup \downarrow y_{23}$, and $\downarrow y'_{12} \cup \{x_3\}$, a contradiction. □

Corollary 6.3. *The following hold:*

(1) $P \cong \bar{2} \oplus \mathbb{N}$;
(2) $L \cong \mathbf{2}^2 \boxplus \mathbb{N}_0$;

(3) $f(n) = \begin{cases} 2 & \text{if } n = 0, \\ 1 & \text{if } 1 \leqslant n \leqslant 2 \end{cases}$

(Figs. 14 and 18).

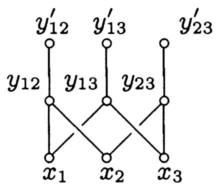

Fig. 17. An impossible scenario ($f(0) = 3$, $f(1) = 2$, $f(2) = 2$)

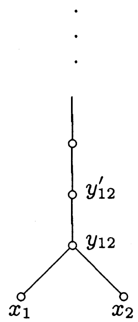

Fig. 18. The poset $\bar{2} \oplus \mathbb{N}$

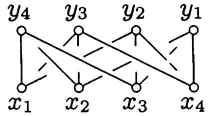

Fig. 19. An impossible scenario ($f(0) = 4,\ f(1) = 3,\ f(2) = 2,\ n = 3,\ f(n) > f(0) - n$)

Proof. By Corollary 6.2, $f(0) = 2$, $f(1) = 1$, and $f(2) = 1$. Assume that, for some $n \geqslant 1$, we have $x_1,\ x_2 \lessdot y \lessdot y' \lessdot \cdots \lessdot y^{(n)}$ where $y^{(i+1)}$ is the unique irreducible upper cover of $y^{(i)}$ in P for $0 \leqslant i < n$.

If $q = y^{(n)}$ in Lemma 4.3, then $A = \emptyset$ and B has 1 element, so $y^{(n)}$ has a unique irreducible upper cover $y^{(n+1)}$ in P.

Let $Q = \{x_1, x_2, y, y', \ldots\}$. Assume for a contradiction that $P \backslash Q \neq \emptyset$. Let $p \in P \backslash Q$ be minimal. By Corollary 4.7, Corollary 4.8(2) and Lemma 6.1, $\mathrm{LC}(p) = \{y^{(n)}\}$ for some $n \geqslant 0$. Thus $p \in \mathrm{Irr}(P)$ so p is the unique irreducible upper cover of $y^{(n)}$, i.e., $p = y^{(n+1)} \in Q$, a contradiction. □

7. f(2) = f(0) − 2

Lemma 7.1. *For* $0 \leqslant n < f(0)$, $f(n) = f(0) - n$.

Proof. The statement is trivial for $0 \leqslant n \leqslant 2$. Assume that $3 \leqslant n < f(0)$ and that $f(k) = f(0) - k$ for $0 \leqslant k < n < f(0)$. By Lemma 4.6, $f(n) \geqslant f(0) - n$. Assume for a contradiction that $f(n) > f(0) - n$.

Then, by the induction hypothesis, there exists for $1 \leqslant i \leqslant n+1$ an element $y_i \in P$ such that $\mathrm{LC}(y_i) = \{x_1, \ldots, x_{n+1}\} \backslash \{x_i\}$ (Fig. 19).

Hence $\downarrow y_1 \cup \{x_1\}$ has *more* than $f(2) = f(0) - 2$ upper covers in L, namely, $\downarrow y_1 \cup \downarrow y_i (2 \leqslant i \leqslant n+1)$ and $\downarrow y_1 \cup \{x_1, x_i\}$ $(n+2 \leqslant i \leqslant f(0))$, a contradiction. □

Corollary 7.2. *There is no element in P whose set of lower covers is a non-empty proper subset of* $\{x_1, \ldots, x_{f(0)}\}$. □

Corollary 7.3. *The inequality* $f(f(0)) \leqslant f(0)$ *holds.*

Proof. Assume for a contradiction that $f(f(0)) \geqslant f(0)+1$. Then by Corollary 7.2, there are at least $f(0)+1$ elements $y \in P$ such that $\mathrm{LC}(y)=\{x_1,\ldots,x_{f(0)}\}$, $y_1,\ldots,y_{f(0)+1}$. So $\downarrow y_1$ has *more* than $f(1)=f(0)-1$ upper covers in L, namely, $\downarrow y_1 \cup \downarrow y_i$ $(2 \leqslant i \leqslant f(0)+1)$, a contradiction. □

Corollary 7.4. *We have* $f(f(0)) \in \{0,1,f(0)\}$.

Proof. Assume not, for a contradiction. Let $k=f(f(0))$. By Corollary 7.3, $2 \leqslant k < f(0)$. Let $y_1,\ldots,y_k$ be the set of all $y \in P$ such that $\mathrm{LC}(y)=\{x_1,\ldots,x_{f(0)}\}$ (Fig. 20). (We are using Corollary 7.2.)

Letting $q=y_1$ in Lemma 4.3, we get $A=\{y_2,\ldots,y_k\}$, so y_1 has $f(1)-(k-1)=f(0)-k$ irreducible upper covers in P, $y_1',\ldots,y_1^{(f(0)-k)}$. Similarly, we have $y_2',\ldots,y_2^{(f(0)-k)}$ (Fig. 21).

Now $\downarrow y_1 \cup \cdots \cup \downarrow y_k$ has *more* than $f(k)=f(0)-k$ (by Lemma 7.1) upper covers in L, namely, $\downarrow y_1^{(i)} \cup \downarrow y_2 \cup \cdots \cup \downarrow y_k$ and $\downarrow y_1 \cup \downarrow y_2^{(i)} \cup \downarrow y_3 \cup \cdots \cup \downarrow y_k$ $(1 \leqslant i \leqslant f(0)-k)$, a contradiction. □

8. $f(2) = f(0)-2$; $f(f(0)) = 0$

Proposition 8.1. *The following hold*:

(1) $P \cong \overline{f(0)}$;
(2) $L \cong \mathbf{2}^{f(0)}$;
(3) $f(n)=f(0)-n$ *if* $0 \leqslant n \leqslant f(0)$.

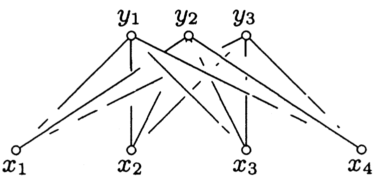

Fig. 20. An impossible scenario ($f(0)=4,\ f(1)=3,\ f(2)=2,\ k=f(4)=3$)

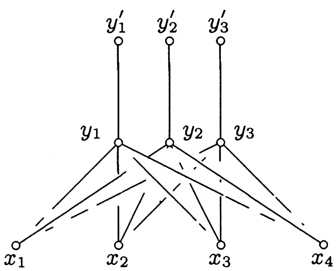

Fig. 21. An impossible scenario ($f(0)=4,\ f(1)=3,\ f(2)=2,\ k=f(4)=3$)

Proof. Assume for a contradiction that $P\backslash\{x_1,\dots,x_{f(0)}\} \neq \emptyset$. Choose a minimal element p in this set; by Corollary 7.2, $\mathrm{LC}(p) = \{x_1,\dots,x_{f(0)}\}$ so $f(f(0)) \neq 0$, a contradiction. The rest follows from Lemma 7.1. □

9. $f(2) = f(0)-2; f(f(0)) = 1$

Lemma 9.1. *We have* $f(0) = 2$.

Proof. Assume for a contradiction that $f(0) \geqslant 3$. There exists a unique element $y \in P\backslash\{x_1,\dots,x_{f(0)}\}$ such that $\mathrm{LC}(y) \subseteq \{x_1,\dots,x_{f(0)}\}$; by Corollary 7.2, $\mathrm{LC}(y) = \{x_1,\dots,x_{f(0)}\}$.

Letting $q = y$ in Lemma 4.3, we get $A = \emptyset$ so y has $f(1) = f(0) - 1$ irreducible upper covers in P, $z_1,\dots,z_{f(0)-1}$ (Fig. 22).

Now let $q = z_1$ in Lemma 4.3; then $A = \{z_2,\dots,z_{f(0)-1}\}$, so z_1 has $f(1) - [(f(0) - 1) - 1] = 1$ irreducible upper cover in P, z_1'. Similarly we have $z_2',\dots,z_{f(0)-1}'$ (Fig. 23).

As $f(0) - 1 \geqslant 2$, $\downarrow z_1 \cup \cdots \cup \downarrow z_{f(0)-1}$ has *more* than $f(f(0) - 1) = 1$ (by Lemma 7.1) upper cover in L, namely, $\downarrow z_1' \cup \downarrow z_2 \cup \cdots \cup \downarrow z_{f(0)-1}$ and $\downarrow z_1 \cup \downarrow z_2' \cup \downarrow z_3 \cup \cdots \cup \downarrow z_{f(0)-1}$, a contradiction. □

10. $f(2) = f(0)-2$ or $f(0) = 2; f(f(0)) = f(0)$

Lemma 10.1. *Assume that, for some* $n \geqslant 1$, *there is a down-set* $Q = \oplus_{i=1}^{n} Y^{(i)}$ *in* P *where, for* $1 \leqslant i \leqslant n$, $Y^{(i)} = \{y_1^{(i)},\dots,y_{f(0)}^{(i)}\} \cong \overline{f(0)}$ *and, for* $1 \leqslant i < n$, $Y^{(i+1)}$ *is the set of all* $y \in P\backslash \downarrow Y^{(i)}$ *such that* $LC(y) \subseteq \downarrow Y^{(i)}$. (*See Fig.* 24)

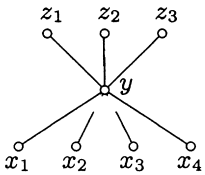

Fig. 22. An impossible scenario ($f(0) = 4,\ f(1) = 3,\ f(2) = 2,\ f(4) = 1$)

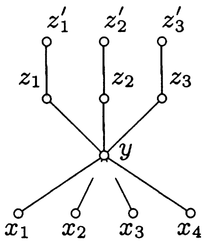

Fig. 23. An impossible scenario ($f(0) = 4,\ f(1) = 3,\ f(2) = 2,\ f(4) = 1$)

Then $Y^{(n+1)} := \{y \in P \setminus \downarrow Y^{(n)} \mid LC(y) \subseteq \downarrow Y^{(n)}\}$ *has exactly* $f(0)$ *elements and* $LC(y) = Y^{(n)}$ *for all* $y \in Y^{(n+1)}$. *(Hence* $Y^{(n+1)}$ *is an antichain.)*

Proof. There exist exactly $f(f(0)) = f(0)$ elements $y \in P \setminus \downarrow Y^{(n)}$ such that $\mathrm{LC}(y) \subseteq \downarrow Y^{(n)}$, $y_1^{(n+1)}, \ldots, y_{f(0)}^{(n+1)}$. We claim that $\mathrm{LC}(y_1^{(n+1)}) \cap Y^{(n)} \neq \emptyset$: Obvious if $n = 1$; otherwise use the assumption on $Y^{(n)}$.

Suppose for a contradiction that $y_1^{(n)} \notin LC(y_1^{(n+1)})$. Then $y_1^{(n)} \not\leq y_1^{(n+1)}$. Hence $\downarrow y_2^{(n)} \cup \cdots \cup \downarrow y_{f(0)}^{(n)}$ has *more* than $f(f(0) - 1) = 1$ (by Lemma 7.1) upper cover in L, namely, the sets $\downarrow Y^{(n)}$ and $\downarrow y_1^{(n+1)} \cup \downarrow y_2^{(n)} \cup \cdots \cup \downarrow y_{f(0)}^{(n)}$, a contradiction. □

Corollary 10.2. *The following hold* (*Fig*. 25):

(1) $P \cong \oplus_{i=1}^{\infty} \overline{f(0)}$;
(2) $L \cong \boxplus_{i=1}^{\infty} \mathbf{2}^{f(0)}$;
(3)

$$f(n) = \begin{cases} f(0) - n & \text{if } 0 \leqslant n < f(0), \\ f(0) & \text{if } n = f(0). \end{cases}$$

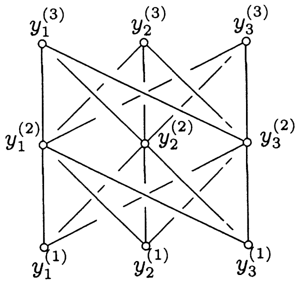

Fig. 24. The situation of Lemma 10.1 ($f(0) = 3$, $f(3) = 3$)

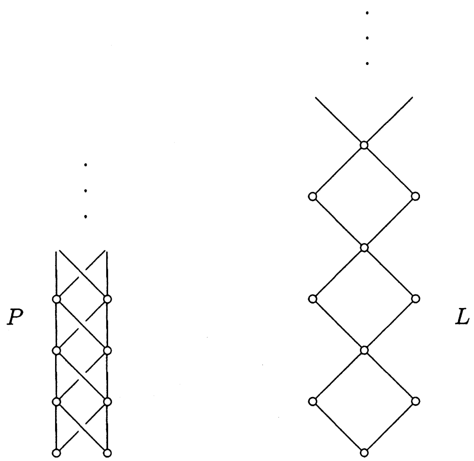

Fig. 25. The situation of Corollary 10.2 ($f(0) = 2$, $f(1) = 1$, $f(2) = 2$)

Proof. Build, by induction, a down-set Q isomorphic to $\oplus_{i=1}^{\infty} Y^{(i)}$ using Lemma 10.1, where for $i \geqslant 1$, $Y^{(i+1)}$ is the $f(0)$-element antichain of all elements $y \in P \setminus \downarrow Y^{(i)}$ such that $\mathrm{LC}(y) \subseteq \downarrow Y^{(i)}$.

Assume for a contradiction that $P \setminus Q \neq \emptyset$; choose $p \in P \setminus Q$ minimal. Then $\mathrm{LC}(p) \subseteq \downarrow Y^{(i)}$ for some $i \geqslant 1$ but $p \in P \setminus \downarrow Y^{(i)}$; hence $p \in Y^{(i+1)}$, a contradiction.

The last part is clear. $\square$

11. The Characterization of Cover Functions

Theorem 11.1. *Let L be a finitary distributive lattice with cover function $f : \mathbb{N}_0 \to \mathbb{N}_0$. Then one of the following holds*:

(1) *for some $k \geqslant 1$, $L \cong \mathbb{N}_0^k$; and for all $n \in \mathbb{N}_0$,*

$$f(n) = \begin{cases} k & \text{if } 0 \leqslant n \leqslant k, \\ * & \text{otherwise;} \end{cases}$$

(2) *for some $k \geqslant 1$, $L \cong Y^k$; and for all $n \in \mathbb{N}_0$, $f(n) = n + k$;*

(3) *for some $k \geqslant 2$, $L \cong \mathbf{1} \oplus \mathbb{N}_0^k$; and for all $n \in \mathbb{N}_0$,*

$$f(n) = \begin{cases} 1 & \text{if } n = 0, \\ k & \text{if } 1 \leqslant n \leqslant k, \\ * & \text{otherwise;} \end{cases}$$

(4) *$L \cong Y \setminus \{0_Y\}$; and for all $n \in \mathbb{N}_0$,*

$$f(n) = \begin{cases} 2 & \text{if } n = 0, \\ n+1 & \text{if } n \geqslant 1; \end{cases}$$

(5) *for some $k \geqslant 0$, $L \cong \mathbf{2}^k$; and for all $n \in \mathbb{N}_0$,*

$$f(n) = \begin{cases} k-n & \text{if } 0 \leqslant n < k, \\ 0 & \text{if } n = k, \\ * & \text{otherwise;} \end{cases}$$

(6) *for some $k \geqslant 2$, $L \cong \boxplus_{i=1}^{\infty} \mathbf{2}^k$; and for all $n \in \mathbb{N}_0$,*

$$f(n) = \begin{cases} k-n & \text{if } 0 \leqslant n < k, \\ k & \text{if } n = k, \\ * & \text{otherwise;} \end{cases}$$

(7) *$L \cong \mathbf{2}^2 \oplus \mathbb{N}$; and for all $n \in \mathbb{N}_0$,*

$$f(n) = \begin{cases} 2 & \text{if } n = 0, \\ 1 & \text{if } 1 \leqslant n \leqslant 2, \\ * & \text{otherwise.} \end{cases}$$

Moreover, the functions listed are cover functions for the corresponding finitary distributive lattices.

Proof. If $f(0) \leqslant f(1)$, we get (1)–(5) by Proposition 4.5.

If $f(0) = 1$ and $f(1) = 0$, then $L \cong \mathbf{2}$ and we get (5).

Hence we may assume that $f(0) \geqslant 2$ and, by Lemma 4.4, that $f(1) = f(0) - 1$.

If $f(2) = f(0)$, then $f(0) = 2$ by Corollary 5.2, so $f(f(0)) = f(0)$. By Corollary 10.2, we have (6).

If $f(2) = f(0) - 1$, then, by Proposition 6.3, we have (7).

By Lemma 4.9, we may assume that $f(2) = f(0) - 2$.

If $f(f(0)) = 0$, we have (5) by Proposition 8.1.

If $f(f(0)) = 1$, we have $f(0) = 2$ by Lemma 9.1, so $f(2) = f(0) - 1$.

By Corollary 7.4, we may assume that $f(f(0)) = f(0)$. By Corollary 10.2, we have (6). $\square$

Thus the problem in Stanley's *Enumerative Combinatorics* is solved.

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