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# Maximal Sublattices of Finite Distributive Lattices. III: A Conjecture from the 1984 Banff Conference on Graphs and Order

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*Abstract.* Let $L$ be a finite distributive lattice. Let $\mathrm{Sub}_0(L)$ be the lattice

$$\{S \mid S \text{ is a sublattice of } L\} \cup \{\varnothing\}$$

and let $\ell_*[\mathrm{Sub}_0(L)]$ be the length of the shortest maximal chain in $\mathrm{Sub}_0(L)$. It is proved that if $K$ and $L$ are non-trivial finite distributive lattices, then

$$\ell_*[\mathrm{Sub}_0(K \times L)] = \ell_*[\mathrm{Sub}_0(K)] + \ell_*[\mathrm{Sub}_0(L)].$$

A conjecture from the 1984 Banff Conference on Graphs and Order is thus proved.

## 1 Motivation

Let $L$ be a finite lattice. Let $\mathrm{Sub}_0(L)$ denote the lattice

$$\{S \mid S \text{ is a sublattice of } L\} \cup \{\varnothing\}$$

ordered by inclusion. (Recall that a lattice or sublattice is by definition non-empty; if $|L| = 1$, we say $L$ is *trivial*.) Let $\ell_*[\mathrm{Sub}_0(L)]$ be the length of the shortest maximal chain in this lattice. Figures 1 through 4 illustrate maximal chains in $\mathrm{Sub}_0(L)$ where $L$ equals $\mathbf{3}$, $\mathbf{2} \times \mathbf{2}$, and $\mathbf{3} \times \mathbf{3}$. (For $n \geq 0$, $\mathbf{n}$ is the $n$-element chain.) We exhibit two maximal chains of $\mathrm{Sub}_0(\mathbf{3}^2)$ of different lengths, one of length 9, one of length 6. How do we know there are not maximal chains that are shorter still?

In [3, Theorem 2(i)], Chen, Koh, and Lee proved the following.

**Theorem 1.1** *Let $m \geq 1$; let $n_1, \ldots, n_m \geq 2$. Then*

$$\ell_*[\mathrm{Sub}_0(\mathbf{n_1} \times \cdots \times \mathbf{n_m})] = \sum_{i=1}^{m} n_i.$$

(Hence the maximal chain of Figure 4 is the shortest possible.)

The papers [1, 6, 7] deal with maximal sublattices of finite distributive lattices.

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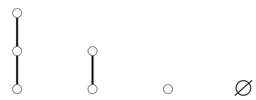

*Figure 1*: A shortest maximal chain in $\mathrm{Sub}_0(\mathbf{3})$; it has length 3.

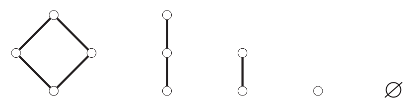

*Figure 2*: A shortest maximal chain in $\mathrm{Sub}_0(\mathbf{2}^2)$; it has length 4.

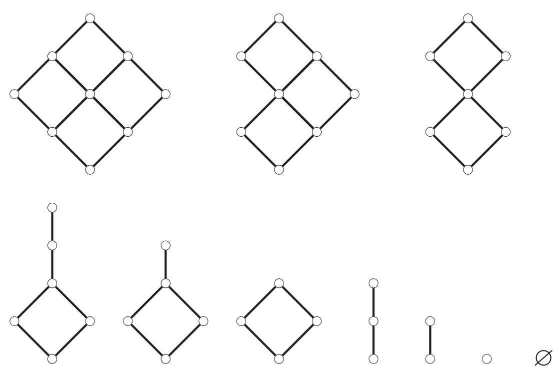

*Figure 3*: A maximal chain in $\mathrm{Sub}_0(\mathbf{3}^2)$; it has length 9.

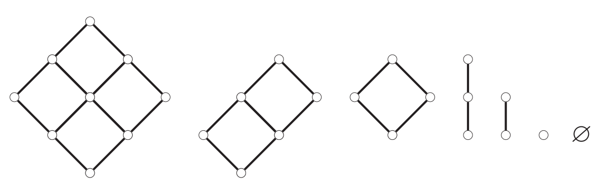

*Figure 4*: Is this a shortest maximal chain in $\mathrm{Sub}_0(\mathbf{3}^2)$?

The following was posed in [3, Problem 1].

***Problem 1.2*** *Let $K$ and $L$ be (non-trivial) finite distributive lattices. Is it always true that $\ell_*[\mathrm{Sub}_0(K \times L)] = \ell_*[\mathrm{Sub}_0(K)] + \ell_*[\mathrm{Sub}_0(L)]$?*

Chen, Koh, and Lee [3] add, "The equality holds if both $L$ and $K$ are products of chains by Theorem 2(i), and up till now we are still unable to find a counterexample."

At the 1984 Banff Conference on Graphs and Order, Koh stated the above as a conjecture [8, p. 554], adding, "It would be nice to prove it if either $L$ or $K$ is a chain." (Note that in neither [3] nor [8] was the word "non-trivial" inserted, though it is clearly needed, as $\ell_*[\mathrm{Sub}_0(\mathbf{1})] = 1$ but $K \times \mathbf{1} \cong K$. Note also that even Figure 2 already shows that Problem 1.2 cannot be solved by naively "splicing" together a maximal chain in $\mathrm{Sub}_0(K)$ with a maximal chain in $\mathrm{Sub}_0(L)$.)

We solve Problem 1.2 below (Theorem 3.3).

## 2 Notation and Basic Results

For notation and terminology not explained here, see [2,4].

Let $P$ be a poset. For $p, q \in P$ such that $p \leq q$, define

$$\downarrow p := \{r \in P \mid r \leq p\}, \quad \overset{\circ}{\downarrow} p := (\downarrow p) \setminus \{p\},$$

$$\uparrow p := \{r \in P \mid r \geq p\}, \quad \overset{\circ}{\uparrow} p := (\uparrow p) \setminus \{p\}.$$

We say $p$ is a *lower cover* of $q$ (and $q$ is an *upper cover* of $p$), denoted $p \lessdot q$, if $p < q$ and $\uparrow p \cap \downarrow q = \{p, q\}$. For $k \geq 0$, let

$$\mathcal{J}_k(P) := \{r \in P \mid r \text{ has exactly } k \text{ lower covers}\},$$

$$\mathcal{M}_k(P) := \{r \in P \mid r \text{ has exactly } k \text{ upper covers}\}.$$

A subset $Q$ of $P$ is a *down-set* of $P$ if $\downarrow r \subseteq Q$ for all $r \in Q$. Let $\mathcal{O}(P)$ denote the bounded distributive lattice of all down-sets of $P$.

Note that sometimes we will deal with two partial orderings at once, for instance, $P$ and $L = \mathcal{O}(P)$. Occasionally, when $Q$ is a subset of a poset $P$, we will give $Q$ the partial ordering inherited from $P$ and call $Q$ a *subposet* of $P$; but sometimes $Q$ will have a different partial ordering. Poset notation relevant to one partial order in cases where there may be confusion will be designated with a subscript, *e.g.*, $a \leq_Q b$ or $\downarrow_L x$. We view the partial order relation $\leq$ as a set of ordered pairs.

Let $P$ and $Q$ be finite posets whose underlying sets are disjoint. Let $P + Q$ be the poset whose underlying set is the disjoint union $P \uplus Q$ and such that for all $r$ and $s$ in $P \uplus Q$ $r \leq_{P+Q} s$ if and only if either $r, s \in P$ and $r \leq_P s$, or else $r, s \in Q$ and $r \leq_Q s$. That is, for all $p \in P$ and $q \in Q$, $p$ and $q$ are incomparable (denoted $p \parallel q$). Note that $\mathcal{O}(P + Q) \cong \mathcal{O}(P) \times \mathcal{O}(Q)$.

Now we come to the first new definition. Let $P$ be a finite poset. A *maximal sublattice sequence for $P$ of size $k$* (where $k \geq 1$) is a sequence of subsets of $P$ (not necessarily subposets)

$$(P_k, P_{k-1}, \ldots, P_2, P_1)$$

such that $P_k = P$, $P_1 = \varnothing$, and, for $1 \leq i < k$, at least one of the following holds (where, for $1 \leq i \leq k$, we let $\leq_i$ denote the partial ordering of $P_i$).

(I) $P_{i+1}$ has a least element $0_{i+1}$ and $P_i$ is the subposet $P_{i+1} \setminus \{0_{i+1}\}$. Let $c_i := 1$.

(II) $P_{i+1}$ has a greatest element $1_{i+1}$ and $P_i$ is the subposet $P_{i+1} \setminus \{1_{i+1}\}$. Let $c_i := 2$.

(III) There exist $x, y \in P_{i+1}$ such that $x \parallel_{i+1} y$, $\overset{\circ}{\downarrow}_{i+1} y \subseteq \overset{\circ}{\downarrow}_{i+1} x$, and $\overset{\circ}{\uparrow}_{i+1} x \subseteq \overset{\circ}{\uparrow}_{i+1} y$; $P_i$ has underlying set $P_{i+1}$ and $\leq_i = \leq_{i+1} \cup \{\, (y, x) \,\}$. Let $c_i := 3$.

(IV) There exist $x \in \mathcal{M}_1(P_{i+1})$ and $y \in \mathcal{J}_1(P_{i+1})$ such that $x <_{i+1} y$ and $P_i$ is the subposet $P_{i+1} \setminus \{x\}$ or $P_{i+1} \setminus \{y\}$. Call $x$ and $y$ the *key* elements and let $c_i = 4$.

We call $(c_{k-1}, \ldots, c_1)$ the *maximal sublattice coding of size* $k - 1$ *associated with* the maximal sublattice sequence.

The point of the above definition is as follows: Birkhoff's theorem says every finite distributive lattice $L$ is isomorphic to $\mathcal{O}(P)$ for some finite poset $P$, which must necessarily be isomorphic to $\mathcal{J}_1(L)$. *Priestley duality* is the dual equivalence between the categories of bounded distributive lattices with $\{0, 1\}$-preserving homomorphisms and Priestley spaces with continuous order-preserving maps. Hence we can describe a maximal $\{0, 1\}$-sublattice (a maximal sublattice containing 0 and 1) $M$ of a finite distributive lattice $L$ by describing the relationship between $P \cong \mathcal{J}_1(M)$ and $Q \cong \mathcal{J}_1(L)$. That relationship must take the form of (III) or (IV). (If $M$ does not contain $0_L$, we get (I); if $M$ does not contain $1_L$, we get (II).)

*Remark.* The description of the "duals" of maximal $\{0, 1\}$-sublattices of finite distributive lattices ((III) and (IV) above) can be gleaned from [1, §3]. The authors do not provide proofs, but state that "Hashimoto [5] was the first to observe that there is a bijective correspondence between the critical pairs of $P$ on one side …[and] with the proper maximal sublattices of $\mathcal{O}(P)$". (The ordered pairs $(y, x)$ in (III) or (IV) satisfy the definition of *criticality* in [1].) We do not find this in [5], although Hashimoto does prove the related theorem [5, Theorem 9.2]. Nevertheless, once one knows what result to aim for, it is routine to prove that the above characterization of maximal $\{0, 1\}$-sublattices is correct. One notes that, except for the beginning, the proof of [7, Theorem 2] applies to any maximal $\{0, 1\}$-sublattice. (This proof itself depends on [6, Theorem 2, Theorem 3], and a converse, which comes from [7, Theorem 1] and the comments at the beginning of [7, §3].) One observes that the element $c$ in the statement of [7, Theorem 2], as the cover of a join-irreducible element of a finite distributive lattice, belongs to $\mathcal{J}_1(L)$ or $\mathcal{J}_2(L)$. In the former case, $M$ is type (IV); in the latter, type (III).

Hence we get the following.

***Lemma 2.1*** *Let $L$ be a finite distributive lattice. Let $P := \mathcal{J}_1(L)$. Then* $\mathrm{Sub}_0(L)$ *has a maximal chain of length $k$ if and only if $P$ has a maximal sublattice sequence of size $k$ if and only if $P$ has a maximal sublattice coding of size $k - 1$.*

*If $L$ is non-trivial and $(P_k, \ldots, P_1)$ is a maximal sublattice sequence, then $k \geq 2$ and $|P_2| = 1$.*

**Proof** If $L = L_k \supsetneqq L_{k-1} \supsetneqq \cdots \supsetneqq L_1 \supsetneqq L_0 = \varnothing$ is a maximal chain in $\mathrm{Sub}_0(L)$, then $L_1$ is trivial. If $L$ is non-trivial, $L_2$ must be $\mathbf{2}$. ■

## 3 Proof of a Conjecture from the 1984 Banff Conference on Graphs and Order

***Proposition 3.1*** *Let $P$ and $Q$ be disjoint, non-empty, finite posets. Let $K := \mathcal{O}(P)$ and let $L := \mathcal{O}(Q)$. Let $k := \ell_*[\mathrm{Sub}_0(K)]$ and let $l := \ell_*[\mathrm{Sub}_0(L)]$; let $j := \ell_*[\mathrm{Sub}_0(K \times L)]$. Then $j \geq k + l$.*

**Proof** Suppose for a contradiction that $j < k+l$. Let $(R_j, R_{j-1}, \dots, R_1)$ be a maximal sublattice sequence for $P + Q$; let $(e_{j-1}, \dots, e_1)$ be the associated maximal sublattice coding and let $\leq_i$ be the partial order of $R_i$ $(1 \leq i \leq j)$. Let

$$\begin{aligned} k' = 1 + \Big| &\{1 \leq i \leq j-1 \mid e_i = 1 \text{ or } 2, \text{ and } R_{i+1} \setminus R_i \subseteq P\} \\ &\cup \{1 \leq i \leq j-1 \mid e_i = 3, \text{ and } \leq_i \setminus \leq_{i+1} \subseteq P \times P\} \\ &\cup \{1 \leq i \leq j-1 \mid e_i = 4, \text{ and both key elements are in } P\}\Big| . \end{aligned}$$

Let $l'$ be the corresponding number for $Q$. Then $k'-1+l'-1 \leq j-1 \leq k-1+l-1$. If $k' \geq k$ and $l' \geq l$, then $k'-1+l'-1 = j-1$. So there would be no $i \in \{1, \dots, j-1\}$ such that $e_i = 3$ and $\leq_{i+1} \setminus \leq_i \subseteq (P \times Q) \cup (Q \times P)$. But this is impossible since $P$ and $Q$ are non-empty, while $R_1 = \varnothing$ and for all $p \in P$ and $q \in Q$, $p \parallel q$ in $R_j = P + Q$. Thus, without loss of generality, $k' < k$.

For $1 \leq i \leq j$, let $P_i$ be the subposet $R_i \cap P$ of $(R_i, \leq_i)$. Except for $k' - 1$ values of $i \in \{1, \dots, j-1\}$, we have $(P_{i+1}, \leq_{i+1}) = (P_i, \leq_i)$ (without loss of generality in case $e_i = 4$). Let the posets corresponding to the exceptions be, in order,

$$\big((\overline{P}_{k'}, \sqsubseteq_{k'}), (\overline{P}_{k'-1}, \sqsubseteq_{k'-1}), \dots, (\overline{P}_1, \sqsubseteq_1)\big) .$$

This is a maximal sublattice sequence for $P$ of size $k' < k$, so, by Lemma 2.1, $\ell_*[\mathrm{Sub}_0(K)] < k$, which is a contradiction. ∎

***Lemma 3.2*** *Let $P$ be a non-empty finite poset. If, for some $k \geq 1$, $P$ has a maximal sublattice coding of size $k-1$, then $P$ has a maximal sublattice coding $(c_{k-1}, \dots, c_1)$ where, for some $a \in \{1, \dots, k-1\}$,*

$$c_{k-1}, \dots, c_{a+1} \in \{3, 4\} \text{ and } c_a, \dots, c_1 \in \{1, 2\}.$$

*Moreover, if the latter's associated maximal sublattice sequence is $(P_k, \dots, P_1)$, then $P_{a+1}, P_a, \dots, P_1$ are chains of size $a, a-1, \dots, 0$, respectively.*

**Proof** If $(d_{k-1}, \dots, d_1)$ is a maximal sublattice coding and, for some

$$i \in \{1, \dots, k-2\}, \quad d_{i+1} \in \{1, 2\}, \quad d_i \in \{3, 4\},$$

then $(d_{k-1}, \dots, d_{i+2}, d_i, d_{i+1}, d_{i-1}, \dots, d_1)$ is also a maximal sublattice coding. By Lemma 2.1, we have $k \geq 2$ and $c_1 \in \{1, 2\}$. ∎

***Theorem 3.3*** *Let $K$ and $L$ be non-trivial finite distributive lattices. Then*

$$\ell_*[\mathrm{Sub}_0(K \times L)] = \ell_*[\mathrm{Sub}_0(K)] + \ell_*[\mathrm{Sub}_0(L)].$$

**Proof** Let $k := \ell_*[\mathrm{Sub}_0(K)]$ and $l := \ell_*[\mathrm{Sub}_0(L)]$. Let $P := \mathcal{J}_1(K)$ and let $Q := \mathcal{J}_1(L)$ (which we can assume to be disjoint). By Lemma 2.1 and Proposition 3.1, we need only show that there is a maximal sublattice sequence for $P+Q$ of size $k+l$.

Applying Lemma 3.2, let $1 \leq a \leq k-1$ be such that $P$ has a maximal sublattice coding $(c_{k-1}, \ldots, c_1)$ where $c_{k-1}, \ldots, c_{a+1} \in \{3,4\}$ and $c_a, \ldots, c_1 \in \{1,2\}$. Let $1 \leq b \leq l-1$ be such that $Q$ has a maximal sublattice coding $(d_{l-1}, \ldots, d_1)$ where $d_{l-1}, \ldots, d_{b+1} \in \{3,4\}$ and $d_b, \ldots, d_1 \in \{1,2\}$.

Now

$$(c_{k-1}, \ldots, c_{a+1}, d_{l-1}, \ldots, d_{b+1}, 4, \ldots, 4, 3, 1, 1),$$

where the 4's displayed appear $a-1+b-1$ times, is a maximal sublattice coding for $P+Q$ of size

$$\begin{aligned}[(k-1)-(a+1)+1] + [(l-1)-(b+1)+1] + (a-1) + (b-1) + 3 \\ = k+l-a-b+a+b-1-1-1-1+3 \\ = k+l-1.\end{aligned}$$

By Lemma 2.1, we are done. (The associated maximal sublattice sequence $(R_{k+l}, \ldots, R_1)$ is such that, by the time the 4's start, we have a disjoint sum of two chains by Lemma 3.2; the 4's reduce the poset to a two-element antichain; the 3 makes it a two-element chain; and the final 1's remove the elements of this chain.) ■

Thus we have proven the conjecture from the 1984 Banff Conference on Graphs and Order.

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